
2


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1

VECTORS IN THREE DIMENSIONS

THREE-DIMENSIONAL VECTORS

TERMINOLOGY

altitude angle
azimuth angle
consistent
cross product
dependent
dot product
Gaussian elimination
homogeneous
inconsistent
independent
inner product
linear equation
non-homogeneous
norm
ordered triple
outer product
polar form
projection
resultant vector
scalar product
system of equations
unit vector
vector product
work

Prior learning

THE ALGEBRA OF VECTORS IN THREE DIMENSIONS

- review the concepts of vectors from Unit 1 and extend to three dimensions including introducing the unit vectors ***i***, ***j*** and ***k*** (ACMSM101)
- prove geometric results in the plane and construct simple proofs in three-dimensions. (ACMSM102)

VECTOR AND CARTESIAN EQUATIONS

- introduce Cartesian coordinates for three-dimensional space, including plotting points and the equations of spheres (ACMSM103)
- use vector equations of curves in two or three dimensions involving a parameter, and determine a 'corresponding' Cartesian equation in the two-dimensional case (ACMSM104)
- determine a vector equation of a straight line and straight-line segment, given the position of two points or equivalent information, in both two and three dimensions (ACMSM105)
- examine the position of two particles each described as a vector function of time, and determine if their paths cross or if the particles meet (ACMSM106)
- use the cross product to determine a vector normal to a given plane (ACMSM107)
- determine vector and Cartesian equations of a plane and of regions in a plane. (ACMSM108)

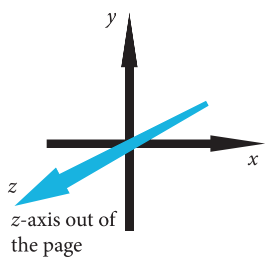
SYSTEMS OF LINEAR EQUATIONS

- recognise the general form of a system of linear equations in several variables, and use elementary techniques of elimination to solve a system of linear equations (ACMSM109)
- examine the three cases for solutions of systems of equations – a unique solution, no solution and infinitely many solutions – and the geometric interpretation of a solution of a system of equations with three variables. (ACMSM110) AC

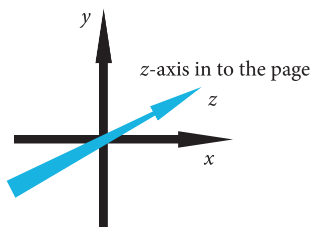
1.01 THREE-DIMENSIONAL CARTESIAN COORDINATES

You need two coordinates for points in two dimensions. For three dimensions, you need three coordinates, so you need a third axis. You can use an x-axis, y-axis and z-axis and write points as **ordered triples** like $P(x, y, z)$.

There are two possible directions in which you can put the z-axis, as shown below for an x-axis across the page and y-axis up the page. The z-axis can come out of the page or go into the page.



Right-handed system



Left-handed system

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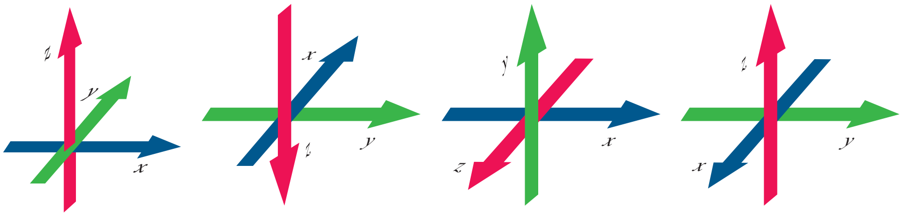
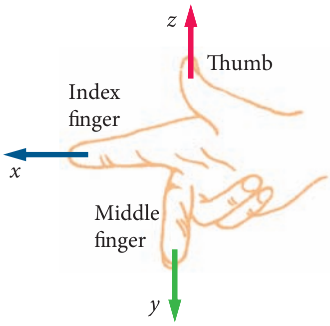
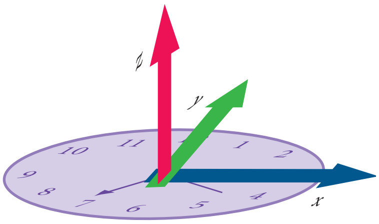
You can see why they are called the left-handed and right-handed systems. It is conventional to use a **right-handed** system. If you curl the fingers of your right hand from the x-axis to the y-axis, your thumb points in the direction of the z-axis.

As long as the relationship between the axes is the same, you can point the x-axis in any direction and still have a right-hand system.



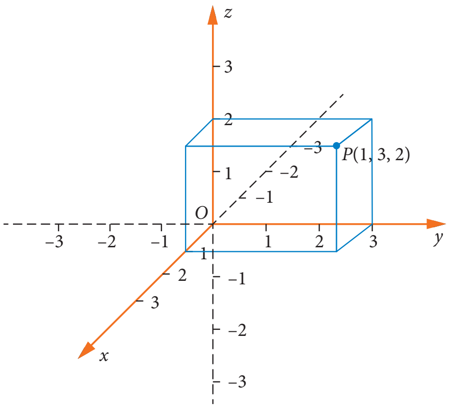
Many people like to make their index ('pointer') finger the x-axis and their middle finger ('tall-man') the y-axis so their thumb is the z-axis like the drawing below. You can also think of it in terms of a clock. If the x-axis is rotated *anticlockwise* to the y-axis, the z-axis comes out of the clock-face. Remember that angles are positive in the anticlockwise direction.





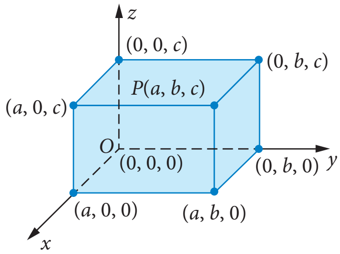
The point P in the diagram on the right has coordinates (1, 3, 2), so it is:

1 unit in the positive direction of the x-axis
3 units in the positive direction of the y-axis
2 units in the positive direction of the z-axis



The point P is intersected by three planes – each one parallel to a coordinate plane. You can imagine P as one vertex of a rectangular box whose faces are each parallel to one of the coordinate planes.

When you plot points in 3D, you'll find a box helpful.

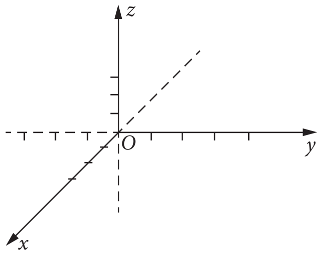


Example 1

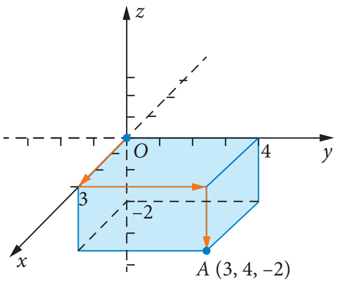
Plot the points $A(3, 4, -2)$ and $B(-1, -3, 1)$.

Solution

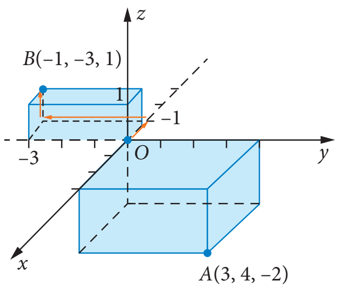
Draw a set of axes for the Cartesian space.



To locate the point $A(3, 4, -2)$, move 3 units in the positive direction of the x-axis, 4 units in the positive direction of the y-axis and 2 units in the negative direction of the z-axis.



To locate the point $B(-1, -3, 1)$, move 1 unit in the negative direction of the x-axis, 3 units in the negative direction of the y-axis and 1 unit in the positive direction of the z-axis.



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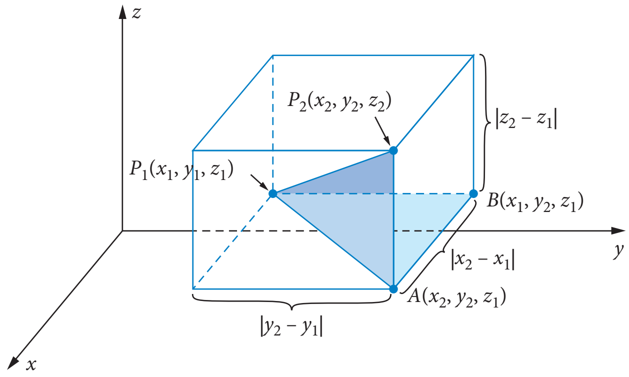
The distance formula to calculate the distance between any two points $P_1(x_1, y_1)$ and $P_2(x_2, y_2)$ in 2D is $d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$

WS
Points in 3D space

You can develop a similar formula for the distance between any two points $P_1(x_1, y_1, z_1)$ and $P_2(x_1, y_1, z_2)$ in 3D.

Construct the points $A(x_2, y_2, z_1)$ and $B(x_1, y_2, z_1)$ as shown in the diagram below.



Then $\triangle P_1BA$ is right-angled, so by Pythagoras' theorem,

$$P_1A^2 = P_1B^2 + BA^2$$
$$= (x_2 - x_1)^2 + (y_2 - y_1)^2.$$

But $\triangle P_1AP_2$ is also right-angled so

$$P_1P_2^2 = P_1A^2 + AP_2^2$$
$$= (x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2$$

So for 3D, you get the formula below.

IMPORTANT

The distance between the points $P_1(x_1, y_1, z_1)$ and $P_2(x_2, y_2, z_2)$ in three dimensions is given by:

$$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$$

It should be noted that if P_1 and P_2 are on the xy-plane, $z_1 = z_2 = 0$, so the 3D distance formula reduces to the distance formula in two dimensions.

The midpoint formula in two dimensions for $P_1(x_1, y_1)$ to $P_2(x_1, y_1)$ is:

$$\left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right)$$

You can extend this to 3D as shown below.

IMPORTANT

3D midpoint formula

The midpoint of the line segment from $P(x_1, y_1, z_1)$ to $Q(x_2, y_2, z_2)$ is:

$$M_{PQ} = \left(\frac{x_1+x_2}{2}, \frac{y_1+y_2}{2}, \frac{z_1+z_2}{2}\right)$$

Example 2

Given the points $P(-1, 3, 1)$ and $Q(3, 4, -2)$,

a find the distance between P and Q

b find the midpoint of PQ.

Solution

a Write the 3D distance formula. $d = \sqrt{(x_2-x_1)^2+(y_2-y_1)^2+(z_2-z_1)^2}$

Substitute for x_1, y_1, etc. $= \sqrt{(3+1)^2+(4-3)^2+(-2-1)^2}$

Evaluate. $= \sqrt{26}$

b Write the midpoint formula. $M_{PQ} = \left(\frac{x_1+x_2}{2}, \frac{y_1+y_2}{2}, \frac{z_1+z_2}{2}\right)$

Substitute for x, y and z values. $= \left(\frac{-1+3}{2}, \frac{3+4}{2}, \frac{1-2}{2}\right)$

Evaluate. $= (1, 3\frac{1}{2}, -\frac{1}{2})$

In the two-dimensional Cartesian system, the equation of a circle is calculated using the fact that any point $P(x, y)$ on the circumference is the same distance r from the centre. If the centre is $C(h, k)$, you get $\sqrt{(x-h)^2+(y-k)^2} = r$ so $(x-h)^2 + (y-k)^2 = r^2$.

In three dimensions, points that are the same distance from a centre $C(h, k, l)$ will not be a line. They will form the surface of a sphere. For a radius r, $r = \sqrt{(x-h)^2+(y-k)^2+(z-l)^2}$

so $(x-h)^2 + (y-k)^2 + (z-l)^2 = r^2$

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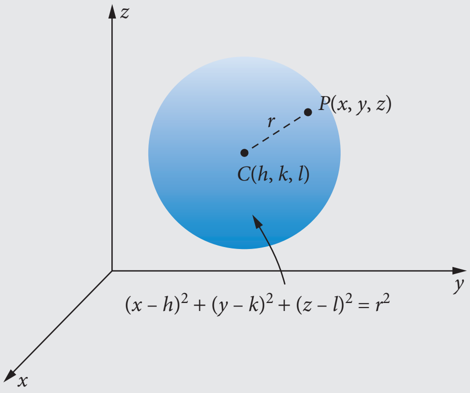
IMPORTANT

Equation of a sphere

The equation of a sphere with centre $C(h, k, l)$ and radius, r, is

$$(x-h)^2+(y-k)^2+(z-l)^2=r^2$$

or $\quad x^2+y^2+z^2+ax+by+cz+d=0.$



Example 3

a Find the equation of a sphere with centre (1, –2, 3) and radius 5.

b Find the radius and centre of a sphere with the equation $x^2+y^2+z^2-6x+8y+4z+4=0$

Solution

a Write the formula.	$(x-h)^2+(y-k)^2+(z-l)^2=r^2$
Substitute in the values.	$(x-1)^2+(y+2)^2+(z-3)^2=5^2$
b Rewrite the equation.	$x^2+y^2+z^2-6x+8y+4z=-4$
Complete the squares.	$(x^2-6x+9)+(y^2+8y+16)+(z^2+4z+4)$ $=-4+9+16+4$
Factorise the squares.	$(x-3)^2+(y+4)^2+(z+2)^2=5^2$
Write the answer.	The centre is (3, –4, –2) and the radius is 5.

In part **a** above, you would not bother expanding the brackets unless you were asked for the expanded form.

INVESTIGATION A sphere on a CAS calculator

Consider the sphere with centre (0, 0, 0) and radius 5. The equation of the sphere is:

$$x^2 + y^2 + z^2 = 5^2$$

In order to graph the sphere using a calculator, you need to rearrange the equation so that it is written in terms of z. Hence, the equation becomes:

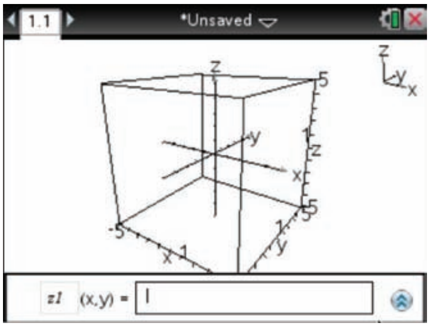
$$z = \pm\sqrt{25-(x^2+y^2)}$$

This needs to be graphed as two separate functions:

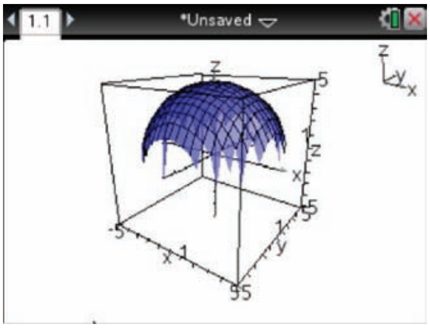
$$z_1 = \sqrt{25-(x^2+y^2)} \text{ and } z_2 = -\sqrt{25-(x^2+y^2)}$$

TI-Nspire CAS

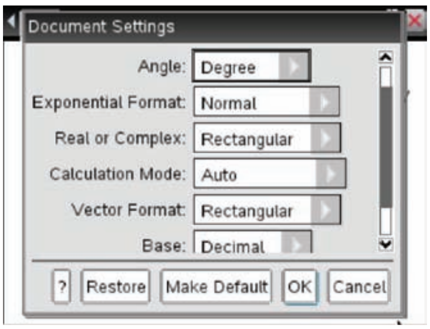
Enter the graphing screen and choose 3D graphing by pressing menu and then selecting 2:View, 3:3d Graphing.



Next enter the equation $\sqrt{25-(x^2+y^2)}$ and press ENTER.
The graph does not look like a hemi-sphere. You need to adjust the calculator settings and the equation to rectify this.



Press the 'battery' symbol to select the calculator settings. Make sure the 'Real or Complex' and 'Vector Format' settings are both 'Rectangular'.



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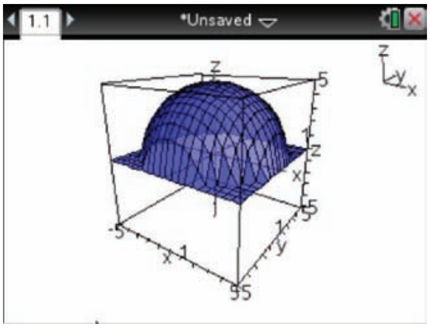
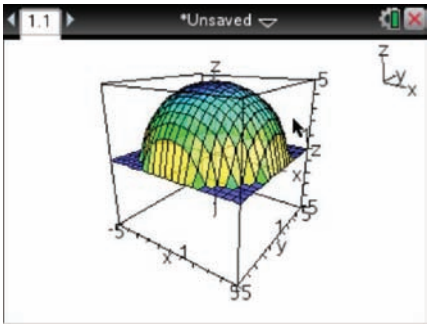
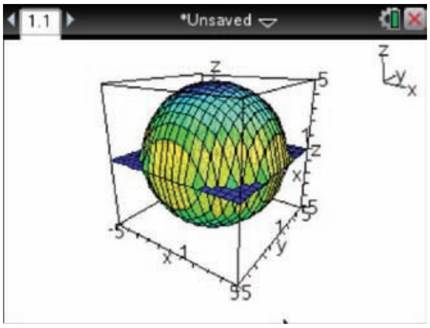
Next you need to edit the equation. Right-click on the graph. You can do this by moving the cursor over the graph and pressing $\boxed{\text{ctrl}}$ $\boxed{\text{menu}}$. Select 6:Edit Relation and change the relation to real $(\sqrt{25-(x^2+y^2)})$ and press OK.
Now the graph looks like a hemi-sphere.



You can change the appearance of the graph to accentuate the 3D nature of the shape by right-clicking on the graph ($\boxed{\text{ctrl}}$ $\boxed{\text{menu}}$) and selecting 8:Colour, then 4:Custom Plot Colour.



Now enter the equation for the other hemi-sphere by pressing $\boxed{\text{ctrl}}$ $\boxed{\text{G}}$ and then enter:
$\boxed{(-)}$ z $\boxed{1}$ $\boxed{(}$ x $\boxed{,}$ y $\boxed{)}$
Press ENTER.
Change the colour of this graph to match the other hemisphere.



ClassPad

3D graphs are not possible.

Experiment with the resulting graph.

- The graph can be rotated using the arrow keys.
- Pressing ‘a’ will auto-rotate the graph.
- The orientation of the graph can be changed by pressing ‘*x*’, ‘*y*’, ‘*z*’ or ‘*o*’.

The magnification of the graph can be increased or decreased using the $\boxed{\times}$ and $\boxed{\div}$ keys respectively.

EXERCISE 1.01 Three-dimensional Cartesian coordinates

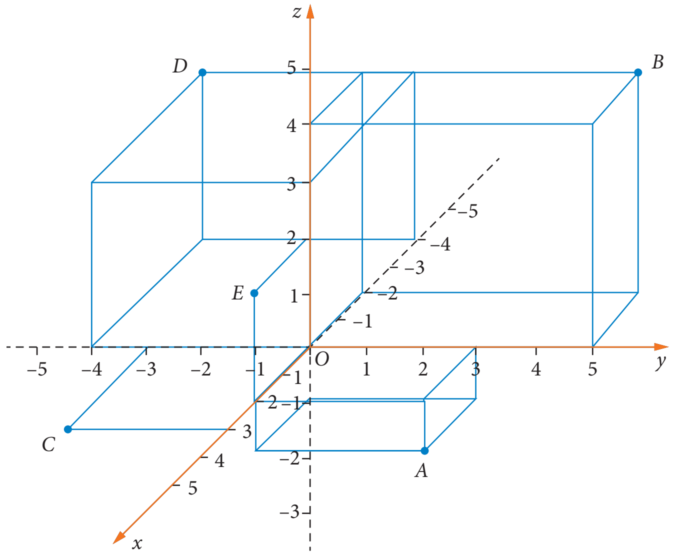
Concepts and techniques

WS Distances, midpoints and spheres

1 Example 1 $P(2, 3, -1)$ and which of the following points are on a line parallel with the z-axis?
A $(2, -3, -1)$ **B** $(4, 3, -1)$ **C** $(0, 3, -1)$ **D** $(2, 3, 4)$ **E** $(2, 0, -1)$

2 The distance between $A(a, b, c)$ and $P(a, d, c)$ is:
A $|c - a|$ **B** $|d - b|$ **C** $|c - b|$ **D** $|d - a|$ **E** $|d - c|$

Use the diagram below to answer questions **3–5**.



3 The point lying on the xy-plane is:
A A **B** B **C** C **D** D **E** E

4 Point E is lying on the:
A xz-plane **B** x-plane **C** xy-plane **D** z-plane **E** yz-plane

5 State the coordinates of A, B, C, D and E.

6 Plot the points A and B in each case below.
a $A(4, -2, 3)$, $B(2, 4, -5)$ **b** $A(3, -2, 1)$, $B(-4, 0, 1)$ **c** $A(0, 0, 0)$, $B(4, -1, -8)$
d $A(1, 1, 0)$, $B(0, 5, -4)$ **e** $A(0, 1, 1)$, $B(1, 0, 0)$ **f** $A(2, 4, -1)$, $B(1, -2, 7)$

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7 If A and B are opposite vertices of a rectangular prism with faces parallel to the xy-, xz- and yz-planes, find the coordinates of the other vertices.

a $A(0, 0, 0)$, $B(2, 4, 3)$ b $A(2, 4, -3)$, $B(-5, 2, 1)$

8 Example 2 For each situation in question **6**, calculate the distance AB.

9 For each situation in question **6**, calculate the midpoint of AB.

10 Given the points A, B, C, D and E in question **5**, calculate the distances

a CA b AE c EB d CB e AB f DA

11 Given the points A, B, C, D and E in question **5**, calculate the midpoints of:

a CA b AE c EB d CB e AB f DA

12 Example 3 In each of the following cases, find the expanded form of the equation of the sphere whose centre (C) and radius (r) is given.

a $C = (0, 0, 0)$, $r = 4$ b $C = (0, 3, 0)$, $r = 5\sqrt{6}$ c $C = (-2, 0, 0)$, $r = 3$

d $C = (3, -1, 2)$, $r = 3$ e $C = (-5, 0, 1)$, $r = \frac{1}{2}$ f $C = (2, -3, 4)$, $r = 7$

13 In each of the following, find the centre and the radius of the sphere.

a $x^2 + y^2 + z^2 - 36 = 0$ b $x^2 + y^2 + z^2 - 10x + 6y - 6z + 34 = 0$

c $4x^2 + 4y^2 + 4z^2 + 8x - 4y - 3 = 0$ d $x^2 + y^2 + z^2 + 4x = 0$

e $x^2 + y^2 + z^2 - 8x + 8z + 16 = 0$ f $x^2 + y^2 + z^2 + 4x - 2y + 2z + 2 = 0$

Reasoning and communication

14 In each case below, P, Q and R are the vertices of a triangle. For each situation, show that $\triangle PQR$ is right-angled and calculate its area.

a $P(2, 3, 1)$, $Q(1, 2, 0)$, $R(0, 1, 2)$ b $P(-3, 2, 4)$, $Q(-2, 1, 6)$, $R(-6, 5, 7)$

15 Find the equation of a sphere whose diameter has endpoints at $A(4, -2, 1)$ and $B(0, 2, -7)$.

16 Find the equation for the set of all points that are equidistant from $(2, -1, 3)$ and $(-1, 5, 1)$.

17 Describe the graph of each of the following equations.

a $z = 3$ b $y = 5$ c $y = 0$

1.02 THREE-DIMENSIONAL VECTORS

A vector from the origin to a point is called a **position vector**. You should remember that it is equal to any vector with the same size and direction. It is easier to convert a vector from one form to another when it is treated as a position vector. You should recall the following about 2D vectors from last year.

IMPORTANT

2-dimensional vectors may be written in **component form**, **polar form** or in terms of the **unit vectors i** and **j** in the x and y directions. For a **position vector a** from the origin to the point $A(a_1, a_2)$, $\mathbf{a} = (a_1, a_2)$, $\mathbf{a} = \begin{bmatrix} a_1 \\ a_2 \end{bmatrix}$, $\mathbf{a} = a_1\mathbf{i} + a_2\mathbf{j}$ or $\mathbf{a} = (r, \theta)$, where

$x = r\cos(\theta)$, $y = r\sin(\theta)$, $r = \sqrt{a_1^2 + a_2^2}$, $\tan(\theta) = \frac{y}{x}$, $\mathbf{i} = (1, 0)$ and $\mathbf{j} = (0, 1)$.

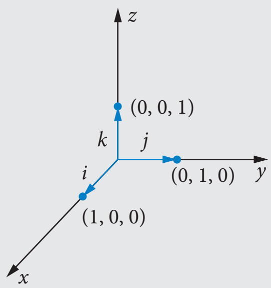
a_1 and a_2 are the **components** of **a**, θ is the **direction** measured anticlockwise from the x-axis and r is the **norm** (**magnitude**) and is also written a or $|\mathbf{a}|$.

The unit vector in the direction of **a** is written as $\hat{\mathbf{a}}$ and is given by $\hat{\mathbf{a}} = \frac{1}{|\mathbf{a}|}\mathbf{a}$.

You should see that the concepts of position vector, component form, norm and unit vectors are easy to extend to three dimensions, but the polar form requires a second angle.

IMPORTANT

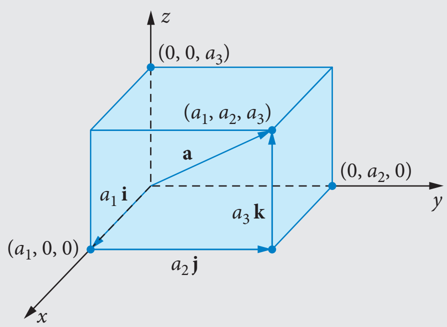
3-dimensional vectors may be written in **component form**, or in terms of the **unit vectors i, j** and **k** in the x, y and z directions.



For a **position vector a = OA** from the origin to the point $A(a_1, a_2, a_3)$, $\mathbf{a} = (a_1, a_2, a_3)$, $\mathbf{a} = \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix}$ or $\mathbf{a} = a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{j}$, where $\mathbf{i} = (1, 0, 0)$, $\mathbf{j} = (0, 1, 0)$ and) $\mathbf{k} = (0, 0, 1)$.

a_1, a_2 and a_3 are the **components** of **a**, and the **norm** (**magnitude**) of **a**, a or $|\mathbf{a}|$, is calculated as $|\mathbf{a}| = \sqrt{a_1^2 + a_2^2 + a_3^2}$.

The unit vector in the direction of **a** is written as $\hat{\mathbf{a}}$ and is given by $\hat{\mathbf{a}} = \frac{1}{|\mathbf{a}|}\mathbf{a}$.



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The properties of addition and multiplication by a scalar for two-dimensional vectors can be extended to three dimensional vectors.

IMPORTANT

If $\mathbf{a} = (a_1, a_2, a_3)$ and $\mathbf{b} = (b_1, b_2, b_3)$ are vectors and c is a scalar:

$$\mathbf{a} + \mathbf{b} = (a_1 + b_1, a_2 + b_2, a_3 + b_3)$$

$$c\mathbf{a} = (ca_1, ca_2, ca_3)$$

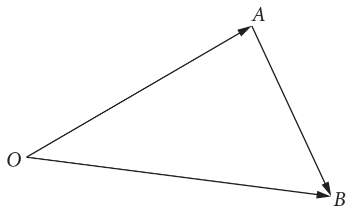
$$\mathbf{0} = (0, 0, 0)$$

$$-\mathbf{a} = -(a_1, a_2, a_3)$$

$$\mathbf{a} - \mathbf{b} = (a_1 - b_1, a_2 - b_2, a_3 - b_3)$$

Parallel vectors are scalar multiples of each other.

When vectors are added, their sum is called the **resultant vector**. Consider the position vectors $\mathbf{a} = \mathbf{OA}$, $\mathbf{b} = \mathbf{OB}$ and the **displacement** vector $\mathbf{AB}$ from A to B.



From the diagram, you can see that

$\mathbf{OA} + \mathbf{AB} = \mathbf{OB}$,

so $\mathbf{AB} = \mathbf{OB} - \mathbf{OA} = \mathbf{b} - \mathbf{a}$.

This is the familiar result that the displacement vector from A to B is the difference between the second position vector and the first position vector. Notice that the diagram assumes nothing about the number of dimensions.

Example 4

Given the position vectors $\mathbf{a} = (2, 1, -3)$ and $\mathbf{b} = (-4, 2, 7)$,

a find $2\mathbf{a} - 3\mathbf{b}$

b show that $(\mathbf{a} - \mathbf{b})$ is parallel to $(\mathbf{b} - \mathbf{a})$

c find $\hat{\mathbf{a}}$.

Solution

a Write the expression using the values of **a** and **b**. $\quad 2\mathbf{a} - 3\mathbf{b} = 2(2, 1, -3) - 3(-4, 2, 7)$

Expand the brackets. $\quad = (4, 2, -6) + (12, -6, -21)$

Subtract components. $\quad = (16, -4, -27)$

State the result. $\quad 2\mathbf{a} - 3\mathbf{b} = (16, -4, -27)$

b Calculate $\mathbf{a} - \mathbf{b}$.
Subtract components.

$$\mathbf{a} - \mathbf{b} = (2, 1, -3) - (-4, 2, 7) = (6, -1, -10)$$

Calculate $\mathbf{b} - \mathbf{a}$.
Subtract components.
Take out -1 as a common factor.

$$\begin{aligned}\mathbf{b} - \mathbf{a} &= (-4, 2, 7) - (2, 1, -3)\\ &= (-6, 1, 10)\\ &= -1(6, -1, -10)\end{aligned}$$

Compare $\mathbf{a} - \mathbf{b}$ and $\mathbf{b} - \mathbf{a}$.

$$\mathbf{a} - \mathbf{b} = -1(\mathbf{b} - \mathbf{a})$$

State the result.

$(\mathbf{a} - \mathbf{b})$ is parallel to $(\mathbf{b} - \mathbf{a})$.

c Write the formula for the unit vector.

$$\hat{\mathbf{a}} = \frac{1}{|\mathbf{a}|}\mathbf{a}$$

Use the rule for the norm of a vector.

$$= \frac{1}{\sqrt{2^2 + 1^2 + (-3)^2}}(2, 1, -3)$$

Evaluate.

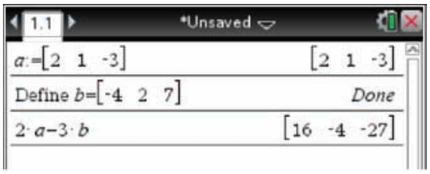
$$= \frac{1}{\sqrt{14}}(2, 1, -3)$$

Write in the standard form.

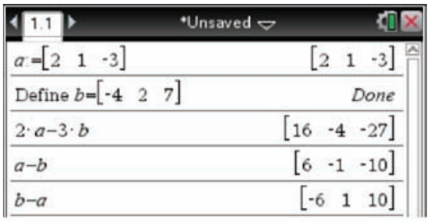
$$= \left(\frac{\sqrt{14}}{7}, \frac{\sqrt{14}}{14}, -\frac{3\sqrt{14}}{14}\right)$$

TI-Nspire CAS

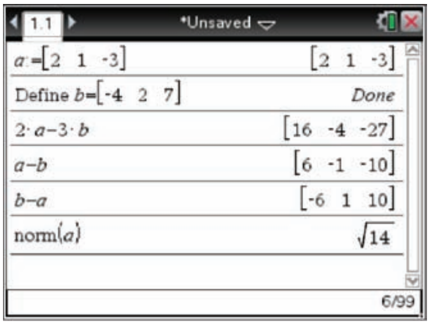
a Enter the vectors as matrices using a [:=] ([ctrl] [templates]) and commas in square brackets or Define and [menu], 7 Matrix & Vector, 1: Create and 1: Matrix.
Perform the required operation.



b Find the differences to compare them.
Clearly $\mathbf{b} - \mathbf{a} = -1(\mathbf{a} - \mathbf{b})$



c Press [menu], 7: Matrix & Vector, 7: Norms and 1: Norm. Put in **a** and press [enter].



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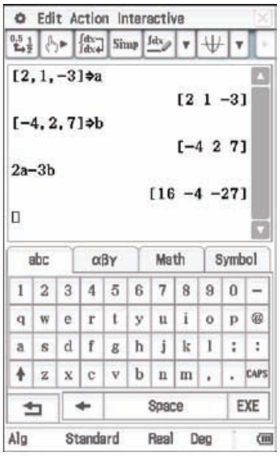
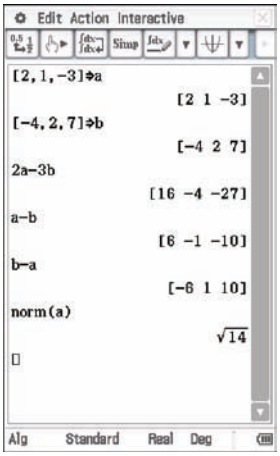
ClassPad

a Ensure that your calculator is set to **Standard** mode.
Enter the vector **a** by pressing [Keyboard], tapping [Math3] and then the square bracket button on this keyboard.
Use the left arrow to place the cursor between the brackets and enter the values of the vector, separated by [,]. Assign this vector to variable name a by tapping [⇒] then [abc] and a, and press [EXE]. Enter the vector **b** in a similar manner.
Perform the required operation.



b Find the differences to compare them.
Clearly $\mathbf{b}-\mathbf{a}=-1(\mathbf{a}-\mathbf{b})$



c Tap **Action** then **Vector** and **norm**, type a then press [)] and [EXE].

You could leave the unit vector in the form $\frac{\sqrt{14}}{14}(2, 1, -3)$. You can also write it as

$$\left(\frac{\sqrt{14}}{7}, \frac{\sqrt{14}}{14}, -\frac{3\sqrt{14}}{14}\right) \approx (0.534, 0.267, -0.802).$$

WS
Resultant and unit vectors in 3D space

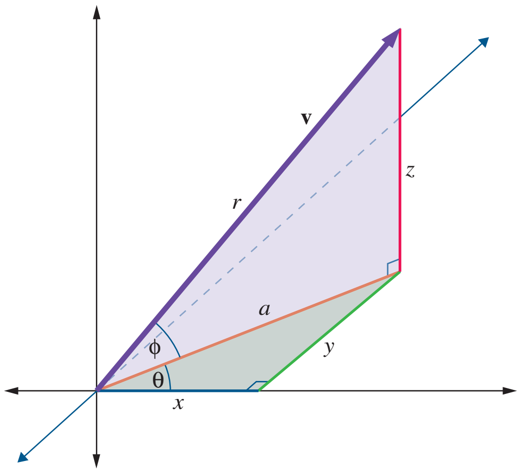
In three dimensions, the polar form of a vector has a magnitude r and two angles, θ and ϕ.

The **azimuth angle** θ lies between 0° and 360° and is the angle from the x-axis to the projection of the vector onto the xy-plane (between x and a).

The **altitude angle** ϕ lies between –90° and 90° and is the angle from the xy-plane to the vector (between a and r).

The azimuth and altitude angles are mainly used in astronomy and navigation.

The polar form $\mathbf{v} = (r, \theta, \phi)$ is related to the rectangular form $\mathbf{v} = (x, y, z)$ as shown in the diagram on the right. Using Pythagoras' theorem, $r = \sqrt{x^2 + y^2 + z^2}$ is just the norm.



In the green triangle,
$x = a \cos(\theta)$ and $y = a \sin(\theta)$

and in the purple triangle
$z = r \sin(\phi)$ and $a = r \cos(\phi)$,

so $x = r \cos(\phi) \cos(\theta)$ and $y = r \cos(\phi) \sin(\theta)$.

Working backwards, $r = \sqrt{x^2 + y^2 + z^2}$ is just the norm, $\sin(\phi) = \frac{z}{r}$ and $\tan(\theta) = \frac{y}{x}$.

IMPORTANT

For the 3D vector $\mathbf{v} = (r, \theta, \phi) = (x, y, z)$:

$x = r \cos(\phi) \cos(\theta)$ $\qquad r^2 = x^2 + y^2 + z^2$

$y = r \cos(\phi) \sin(\theta)$ $\qquad \tan(\theta) = \frac{y}{x}$

$z = r \sin(\phi)$ $\qquad \sin(\phi) = \frac{z}{r}$

and $r = |\mathbf{v}| = \sqrt{x^2 + y^2 + z^2}$

Alamy/David Askham

Example 5

Express the following as linear combinations of **i**, **j** and **k**.

a (2, −7, 3)

b $\begin{bmatrix} -5 \\ 0 \\ 4 \end{bmatrix}$

c A vector of magnitude 5, azimuth 210° and altitude 66°.

Solution

a Write as multiples of **i**, **j** and **k**. $(2, -7, 3) = 2\mathbf{i} + (-7)\mathbf{j} + 3\mathbf{k}$

Simplify. $= 2\mathbf{i} - 7\mathbf{j} + 3\mathbf{k}$

b Write as multiples of **i**, **j** and **k**. $\begin{bmatrix} -5 \\ 0 \\ 4 \end{bmatrix} = -5\mathbf{i} + 0\mathbf{j} + 4\mathbf{k}$

Simplify. $= -5\mathbf{i} + 4\mathbf{k}$

c Write the given information. $r = 5$, $\theta = 210°$ and $\phi = 66°$

Write the formula for x. $x = r\cos(\phi)\cos(\theta)$

Substitute and evaluate. $= 5\cos(66°)\cos(210°)$

≈ -1.761

Write the formula for y. $y = r\cos(\phi)\sin(\theta)$

Substitute and evaluate. $= 5\cos(66°)\sin(210°)$

≈ -1.017

Write the formula for z. $z = r\sin(\phi)$

Substitute and evaluate. $= 5\sin(66°)$

≈ 4.568

Write in terms of **i**, **j** and **k**. Vector $\approx -1.761\mathbf{i} - 1.017\mathbf{j} + 4.568\mathbf{k}$

When you add three-dimensional vectors, it is difficult to calculate the resultant using the polar form. It is difficult to correctly visualise and apply trigonometry directly to the triangles of vectors because they are three-dimensional. It is far easier to convert the vectors to component form before adding.

Example 6

If **a** has magnitude 3, azimuth 47° and altitude −60°, and **b** has magnitude 6, azimuth 223° and altitude −40°, find **a** + **b**.

Solution

Calculate x for **a**.

$$x = 3\cos(-60°)\cos(47°) \approx 1.023$$

Calculate y for **a**.

$$y = 3\cos(-60°)\sin(47°) \approx 1.097$$

Calculate z for **a**.

$$z = 3\sin(-60°) \approx -2.598$$

Write **a** in terms of components.

$$\mathbf{a} \approx (1.023, 1.097, -2.598)$$

Calculate x for **b**.

$$x = 6\cos(-40°)\cos(223°) \approx -3.361$$

Calculate y for **b**.

$$y = 6\cos(-40°)\sin(223°) \approx -3.135$$

Calculate z for **b**.

$$z = 6\sin(-40°) \approx -3.857$$

Write **b** in terms of the components.

$$\mathbf{b} \approx (-3.361, -3.135, -3.857)$$

Calculate **a** + **b**.

$$\mathbf{a}+\mathbf{b} \approx (1.023, 1.097, -2.598) + (-3.361, -3.135, -3.857)$$

Simplify.

$$= (-2.338, -2.038, -6.455)$$

Write the components of **a** + **b**.

$$x \approx -2.338,\ y \approx -2.038,\ z \approx -6.455$$

Calculate the magnitude of **a** + **b**.

$$r \approx \sqrt{(-2.338)^2 + (-2.038)^2 + (-6.455)^2} \approx 7.161$$

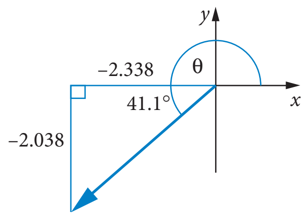
Calculate the azimuth of **a** + **b**.

$$\tan(\theta) \approx \frac{-2.038}{-2.338}$$

Use your calculator and round off.

$$\theta \approx 41.1°$$

You can check this by making a rough sketch. From the sketch, you can see that $\theta \approx 221.1°$ when the quadrant is taken into account.



State the azimuth of **a** + **b**.

Azimuth of **a** + **b** ≈ 221.1°

Calculate ϕ, the altitude of **a** + **b**.

$$\sin(\phi) \approx \frac{-6.455}{7.161}$$

Use your calculator and round off.

$$\phi \approx -64.3°$$

State the result.

The resultant vector has magnitude about 7.161, azimuth 221.1° and altitude −64.3°.

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The last part of Example **6** shows how to convert from the component form to the polar form. The calculation of θ is subject to the same problems you encountered in the two-dimensional case. In order to find the correct angle, you can draw a triangle in the same way as was done in that case. It is not necessary to include the z-component in the triangle because this is accounted for in the calculation of ϕ. In fact, for the altitude, the sign of ϕ is determined solely by the sign of z.

WS
Vectors in polar form

EXERCISE 1.02 Three-dimensional vectors

Concepts and techniques

1 Example 4 If the position vectors of A and B are $\mathbf{a} = (1, 1, -1)$ and $\mathbf{b} = (3, -2, -4)$, then $|\mathbf{AB}|$ is equal to:

A $\sqrt{3}$ B 4 C $\sqrt{22}$ D $\sqrt{26}$ E $\sqrt{29}$

2 Given that $\mathbf{a} = (-2, 1, 3)$, $\mathbf{b} = (3, -1, 2)$ and $\mathbf{c} = (4, -2, -3)$, calculate:

a $2\mathbf{a} + 3\mathbf{b}$ b $\mathbf{b} - 4\mathbf{c}$ c $2\mathbf{a} - \mathbf{b} + \mathbf{c}$

d $2\mathbf{a} - \mathbf{b} + 2\mathbf{c}$ e $2\mathbf{c} + 3\mathbf{b} - 4\mathbf{a}$ f $5\mathbf{a} - 3\mathbf{c} + 2\mathbf{b}$

3 Calculate the norm of the resultant vector in each part of question **2**.

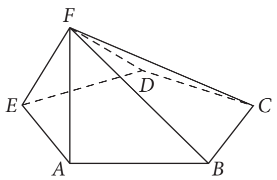
4 Given that $\mathbf{a} = (4, -2, 6)$, $\mathbf{b} = (9, -12, -6)$, $\mathbf{c} = (6, -3, 9)$, $\mathbf{d} = (-6, 8, 4)$ and $\mathbf{e} = (1, -2, -1)$, determine whether the following pairs of vectors are parallel.

a $\frac{3}{2}\mathbf{a}$ and $3\mathbf{c}$ b $(\mathbf{a} + 2\mathbf{e})$ and $(\mathbf{c} + \mathbf{b})$ c $3\mathbf{b}$ and $-2\mathbf{d}$

d $\frac{3}{2}\mathbf{b}$ and $2\mathbf{d}$ e $(\mathbf{a} - \mathbf{c})$ and $(\mathbf{c} - \mathbf{a})$ f $(2\mathbf{a} - \mathbf{b})$ and $(3\mathbf{e} + 2\mathbf{d})$

5 *ABCDE* is a regular pentagon and F is a point not in the plane of the pentagon. The position vectors of A, B, C, D, E and F are $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$, $\mathbf{d}$, $\mathbf{e}$ and $\mathbf{f}$ respectively. Express each of the following vectors in terms of $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$, $\mathbf{d}$, $\mathbf{e}$ and $\mathbf{f}$.

a **AB** b **AC** c **DF** d **FC** e **EB**



6 CAS The points $A(5, 3, 6)$, $B(3, -2, 1)$, $C(6, -2, -4)$, $D(-2, -5, 4)$, and $E(-4, -3, -5)$ have position vectors $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$, $\mathbf{d}$ and $\mathbf{e}$. Use the position vectors to find the following displacement vectors in both component and polar form.

a **BE** b **CD** c **AB** d **BC** e **EA**

7 CAS Find the norms of the following vectors.

a $(8, -5, -7)$ b $(7, -3, 4)$ c $(-2, 5, 9)$

8 Find the unit vector in the direction of each of the following, correct to 4 decimal places.

a $(4, -7, 4)$ b $(-6, 2, -3)$ c $(-5, -3, 4)$

9 If $A = (2, -3, 5)$ and $B = (-4, 2, -3)$, then **AB** is equal to:

A $-2\mathbf{i} - \mathbf{j} + 2\mathbf{k}$ B $3\mathbf{i} - 2\mathbf{j} + 4\mathbf{k}$ C $-6\mathbf{i} + 5\mathbf{j} - 8\mathbf{k}$

D $6\mathbf{i} - 5\mathbf{j} - 5\mathbf{k}$ E $-6\mathbf{i} + 5\mathbf{j} + 3\mathbf{k}$

10 Example 5 Express the following vectors as linear combinations of $\mathbf{i}$, $\mathbf{j}$ and $\mathbf{k}$ and state the norm of each vector.

a $(2, -3, 4)$　b $(-3, -2, 4)$　c $(3, 1, -5)$

d $(-1, 4, -4)$　e $(-3, -1, -2)$　f $\begin{bmatrix} -6 \\ 2 \\ 5 \end{bmatrix}$

g $\begin{bmatrix} 4 \\ -7 \\ 2 \end{bmatrix}$　h $\begin{bmatrix} 1 \\ 6 \\ 4 \end{bmatrix}$　i $\begin{bmatrix} -5 \\ -3 \\ 2 \end{bmatrix}$

j $\begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$　k $(10, 294°, -27°)$　l $(4, 225°, 32°)$

m $(11, 65°, 84°)$　n $\left(16, \frac{5\pi}{3}, -\frac{\pi}{6}\right)$　o $(8, 119°, -70°)$

p Magnitude 50, azimuth 110°, altitude 25°

q Magnitude 80, azimuth $\frac{5\pi}{4}$, altitude $-\frac{\pi}{3}$

11 Convert the following vectors to polar form.

a $(3, -5, -4)$　b $(-4, -2, 4)$　c $\begin{bmatrix} 3 \\ 7 \\ 4 \end{bmatrix}$　d $\begin{bmatrix} -2 \\ -8 \\ -5 \end{bmatrix}$

12 Convert the following vectors to component form.

a $(10, 26°, 75°)$　b $(8, 120°, -46°)$　c $\left(20, \frac{11\pi}{6}, \frac{\pi}{3}\right)$

d 28 at azimuth 210° and altitude −50°

e 32 at azimuth $\frac{3\pi}{4}$ and altitude $\frac{\pi}{6}$

13 Example 6 If $\mathbf{OA} = \mathbf{a} + 3\mathbf{b}$ and $\mathbf{OB} = 5\mathbf{a} - 2\mathbf{b}$, then $\mathbf{AB}$ is equal to:

A $4\mathbf{a} - 5\mathbf{b}$　B $4\mathbf{a} - \mathbf{b}$　C $6\mathbf{a} - 5\mathbf{b}$

D $2\mathbf{a} - 3\mathbf{b}$　E $6\mathbf{a} - 5\mathbf{b}$

14 The position vectors $\mathbf{a} = (12, 30°, -60°)$ and $\mathbf{b} = (15, 150°, 45°)$. Find each of the following in polar form.

a $\mathbf{a} + \mathbf{b}$　b $\mathbf{a} - \mathbf{b}$　c $\mathbf{a} + 2\mathbf{b}$

d $2\mathbf{a} - \mathbf{b}$　e $3\mathbf{a} + 2\mathbf{b}$

15 If $\mathbf{p}$ has norm 30, azimuth $\frac{7\pi}{6}$ and altitude $\frac{\pi}{3}$, and $\mathbf{q}$ has norm 25, azimuth $\frac{7\pi}{4}$ and altitude $-\frac{\pi}{6}$, find each of the following.

a $\mathbf{p} + \mathbf{q}$　b $\mathbf{q} - \mathbf{p}$　c $3\mathbf{p} - \mathbf{q}$

d $2\mathbf{q} - \mathbf{p}$　e $2\mathbf{p} - \mathbf{q}$

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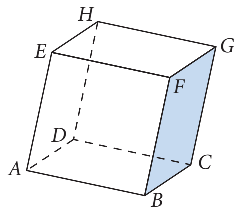
Reasoning and communication

16 *ABCDEFGH* is a cube. The position vectors of *A*, *B*, *C* and *F* are **a**, **b**, **c** and **f** respectively. Find the position vectors of the other vertices and the centre of the cube in terms of **a**, **b**, **c** and **f**.



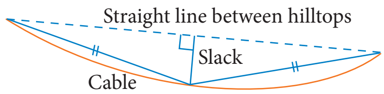
17 An airbus is flying north-east at a height of 15 000 m and a speed of 500 km/h. It is climbing at an angle of 12°. Unfortunately, it flies into some turbulence and drops 1500 m in a time of 1 minute while moving 2000 m to the left of its path. Consider the initial position of the airbus as the origin and calculate the following.

a The vector for its flight displacement through the air in that minute.

b The vector for its movement by the air.

c The vector for its true displacement in that minute.

18 A hilltop that is 400 m above sea level is at map coordinates (50, 70) and another hilltop of elevation 320 m is at coordinates (80, 35). The map coordinates are 10 m apart. Use position and displacement vectors to find the minimum length of a power transmission cable between the hills. The cable requires slack of at least 5% of its length to prevent it snapping in high winds. Although the cable hangs in a curve, the length of the slack cable can be calculated as if it is two straight lines, as shown in the diagram.



19 A submarine travels in a straight line. After descending to a depth of 400 m, its position is 600 m south and 300 m east of its starting point. Find the sub's direction of travel, the angle of descent and the actual distance travelled.

20 A yacht tacking into the wind has a sail area of 40 m^2. The wind pressure on the sail is 500 N/m^2, and the sail deflects the wind up by an angle of 15° and towards the rear of the yacht by an angle of 20°. The wind is originally horizontal at an angle of 60° to the yacht.

a Find the new direction of the wind.

b Find the total force exerted by the wind on the sail.

c Find the components of the force acting on the wind.

d Find the resulting forward force on the yacht (as the negative of the wind change).

e Find the component of the force of the wind tending to sink the yacht.

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1.03 THE SCALAR PRODUCT

The **scalar product** can be found for a 3D vector in a similar way to 2D vectors.

IMPORTANT

OA and **OB** are position vectors such that **OA** = **RS** and **OB** = **PQ**.

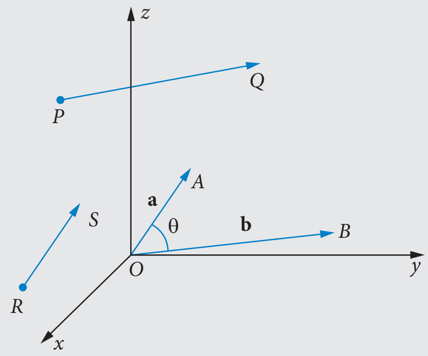
The scalar product of **PQ** and **RS** is given by

$\mathbf{PQ} \cdot \mathbf{RS} = \mathbf{a} \cdot \mathbf{b} = |\mathbf{a}||\mathbf{b}| \cos(\theta) = ab \cos(\theta)$

The scalar product is also called the **inner product** or **dot product**. It is a scalar (real number), not a vector. It may be calculated from vectors in component form as

$\mathbf{a} \cdot \mathbf{b} = (a_1, a_2, a_3) \cdot (b_1, b_2, b_3) = a_1b_1 + a_2b_2 + a_3b_3$

where $\mathbf{a} = (a_1, a_2, a_3)$ and $\mathbf{b} = (b_1, b_2, b_3)$.



i and **j** are perpendicular so
$\mathbf{i} \cdot \mathbf{j} = |\,\mathbf{i}\,| \times |\,\mathbf{j}\,| \times \cos(90°) = 1 \times 1 \times 0 = 0.$

k and **k** are parallel, so
$\mathbf{k} \cdot \mathbf{k} = |\,\mathbf{k}\,| \times |\,\mathbf{k}\,| \times \cos(0°) = 1 \times 1 \times 1 = 1.$

You can set out all the possible products of $a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k}$ and $b_1\mathbf{i} + b_2\mathbf{j} + b_3\mathbf{k}$ in a square.

	$b_1\mathbf{i}$	$b_2\mathbf{j}$	$b_3\mathbf{k}$
$a_1\mathbf{i}$	$a_1\mathbf{i} \cdot b_1\mathbf{i}$	$a_1\mathbf{i} \cdot b_2\mathbf{j}$	$a_1\mathbf{i} \cdot b_3\mathbf{k}$
$a_2\mathbf{j}$	$a_2\mathbf{j} \cdot b_1\mathbf{i}$	$a_2\mathbf{j} \cdot b_2\mathbf{j}$	$a_2\mathbf{j} \cdot b_3\mathbf{k}$
$a_3\mathbf{k}$	$a_3\mathbf{k} \cdot b_1\mathbf{i}$	$a_3\mathbf{k} \cdot b_2\mathbf{j}$	$a_3\mathbf{k} \cdot b_3\mathbf{k}$

In the square, it is the products on the inside that are not zero. This is why it is called the inner product.

IMPORTANT

Properties of the scalar product

1 $\mathbf{a} \cdot \mathbf{b} = |\mathbf{a}||\mathbf{b}| \cos(\theta)$, where θ is the angle between **a** and **b**.

2 The scalar product is a real number, *not a vector*.

3 The scalar product is commutative: $\mathbf{a} \cdot \mathbf{b} = \mathbf{b} \cdot \mathbf{a}$

4 The scalar product is distributive over vector addition:

$\mathbf{a} \cdot (\mathbf{b} + \mathbf{c}) = \mathbf{a} \cdot \mathbf{b} + \mathbf{a} \cdot \mathbf{c}$ and $\mathbf{a} \cdot (\mathbf{b} - \mathbf{c}) = \mathbf{a} \cdot \mathbf{b} - \mathbf{a} \cdot \mathbf{c}$

5 $m(\mathbf{a} \cdot \mathbf{b}) = (m\mathbf{a}) \cdot \mathbf{b} = \mathbf{a} \cdot (m\mathbf{b})\ \forall\ m \in \boldsymbol{R}$ and vectors **a**, **b**.

6 $\mathbf{a} \cdot \mathbf{b} = 0 \Rightarrow \mathbf{a} = \mathbf{0}$ or $\mathbf{b} = \mathbf{0}$ or **a** and **b** are perpendicular.

In terms of components:

7 $\mathbf{i} \cdot \mathbf{i} = \mathbf{j} \cdot \mathbf{j} = \mathbf{k} \cdot \mathbf{k} = 1, \mathbf{i} \cdot \mathbf{j} = \mathbf{j} \cdot \mathbf{k} = \mathbf{k} \cdot \mathbf{i} = 0$

8 If $\mathbf{a} = a_1\mathbf{i} + a_2\,\mathbf{j} + a_3\mathbf{k}$ and $\mathbf{b} = b_1\mathbf{i} + b_2\,\mathbf{j} + b_3\mathbf{k}$, then $\mathbf{a} \cdot \mathbf{b} = a_1b_1 + a_2b_2 + a_3b_3$.

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The definition of the scalar product in terms of magnitudes and the angle between vectors can be used in reverse to find the angle.

Since $\mathbf{a} \cdot \mathbf{b} = |\mathbf{a}|\,|\mathbf{b}| \cos(\theta)$,

$$\cos(\theta) = \frac{\mathbf{a} \cdot \mathbf{b}}{|\mathbf{a}||\mathbf{b}|}.$$

You can use a CAS calculator to calculate scalar products. This is no problem when the coordinates are in component form. However, CAS calculators use **spherical coordinates** for the polar form. Instead of the altitude, they use the **polar angle**, which is measured anticlockwise from the z-axis. In this book, the symbol ψ will be used for the polar angle to avoid confusion. Unfortunately, the polar angle is also shown as ϕ or φ, so there can be confusion. Clearly $\phi + \psi = 90°$.

Example 7

a Find the scalar product of vectors of magnitudes 5 and 6 at an angle of 60° to each other.

b For $\mathbf{a} = 2\mathbf{i} - 3\mathbf{j} + 6\mathbf{k}$ and $\mathbf{b} = -7\mathbf{i} + \mathbf{j} + 5\mathbf{k}$, find $\mathbf{a} \cdot \mathbf{b}$.

c The vector $\mathbf{a}$ has magnitude 12, azimuth angle 102° and altitude 47°; vector $\mathbf{b}$ has magnitude 8, azimuth angle 58° and altitude −15°. Find $\mathbf{a} \cdot \mathbf{b}$.

Solution

a Write the formula.

$\mathbf{a} \cdot \mathbf{b} = |\mathbf{a}||\mathbf{b}| \cos(\theta)$

Substitute the values.

$= 5 \times 6 \times \cos(60°)$

$= 15$

b Write the formula.

$\mathbf{a} \cdot \mathbf{b} = a_1 b_1 + a_2 b_2 + a_3 b_3$

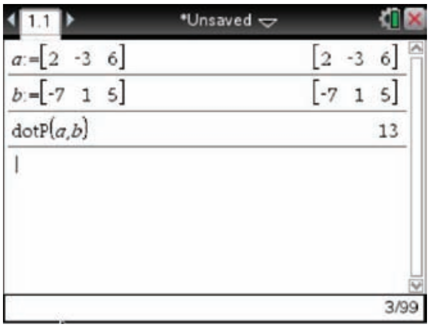
Substitute in the values.

$= 2 \times (-7) + (-3) \times 1 + 6 \times 5$

$= 13$

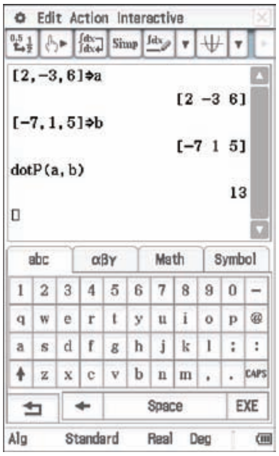
TI-Nspire CAS

Set up the vectors.
Use menu, 7:Matrix & Vector, C:Vector and 3:Dot Product, or type dotP.



ClassPad

Set up the vectors. One way of doing this is to use the [] button available under Keyboard Math3, with elements of a vector separated by a comma.
Tap ⇒, then a vector name from the keyboard under abc.
Tap **Action Vector dotP**, then enter a , b) and press EXE.



c Use $x = r\cos(\theta)\cos(\phi)$,
$y = r\sin(\theta)\cos(\phi)$ and
$z = r\sin(\phi)$
to express the vectors in component form.

$\mathbf{a} = (12, 102°, 47°)$
$\approx (-1.702, 8.005, 8.776)$
$\mathbf{b} = (8, 58°, -15°)$
$\approx (4.095, 6.553, -2.071)$

Calculate the product.

$$\begin{aligned}\mathbf{a}\cdot\mathbf{b} &\approx (-1.702, 8.005, 8.776)\cdot(4.095, 6.553, -2.071)\\ &= -1.702\times 4.095 + 8.005\times 6.553\\ &\quad + 8.776\times(-2.071)\\ &\approx 27.320\end{aligned}$$

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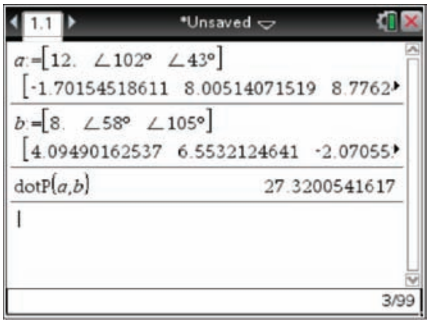
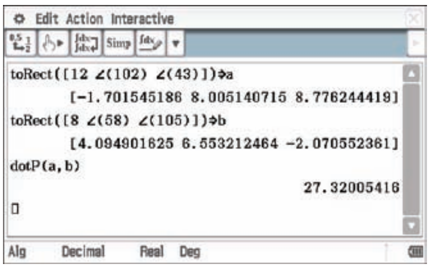
Use $\phi + \psi = 90°$ to find the polar angles. Set up the vectors using the angle and degree symbols from the symbol menu (ctrl). Force approximations by inserting decimal points.

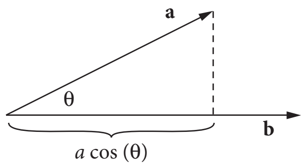


ClassPad

Use $\phi + \psi = 90°$ to find the polar angles. Ensure that your calculator is set to **Decimal** mode. To express vector **a** in component form, first tap **Action Vector toRect**. Enter the vector using the angle symbol ∠ found under Keyboard Math3 and assign the result to **a** as shown. Follow a similar process for the vector **b**. Calculate the dot product of **a** and **b** as before.



You found projections in two dimensions last year. Remember that the projection of one vector on another is the component of the vector in that direction. If **a** and **b** have the same initial point, and the angle between them is θ, then the projection of **a** in the direction of **b**, namely **p**, has magnitude $a \cos(\theta)$.



$p = a\cos(\theta)$ or $|\mathbf{p}| = |\mathbf{a}|\cos(\theta)$

The unit vector in the direction of **b** is $\hat{\mathbf{b}}$, so

$$\begin{aligned}\mathbf{p} &= |\mathbf{a}|\cos(\theta) \times \hat{\mathbf{b}} \\ &= |\mathbf{a}| \times 1 \times \cos(\theta) \times \hat{\mathbf{b}}\end{aligned}$$

Replace 1 by $|\hat{\mathbf{b}}|$.

$$\begin{aligned}&= |\mathbf{a}| \times |\hat{\mathbf{b}}| \times \cos(\theta) \times \hat{\mathbf{b}} \\ &= (\mathbf{a} \cdot \hat{\mathbf{b}})\,\hat{\mathbf{b}}\end{aligned}$$

IMPORTANT

The magnitude p of the projection **p** of **a** on **b** can be calculated by the dot product of **a** and the unit vector in the direction of **b**. Since the direction of a projection is fixed, it is often referred to by only its magnitude.

$$p = \mathbf{a} \cdot \hat{\mathbf{b}} \text{ and } \mathbf{p} = (\mathbf{a} \cdot \hat{\mathbf{b}})\,\hat{\mathbf{b}}$$

This can be rewritten as

$$p = \mathbf{a} \cdot \hat{\mathbf{b}} \times \frac{|\mathbf{b}|}{|\mathbf{b}|} = \frac{(\mathbf{a} \cdot \mathbf{b})}{|\mathbf{b}|}$$

so

$$\mathbf{p} = \frac{(\mathbf{a} \cdot \mathbf{b})}{|\mathbf{b}|}\,\hat{\mathbf{b}}.$$

The scalar product can also be used to find the projection of a vector in the direction of another vector, to find a projection in a particular direction, or to show that vectors are perpendicular.

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Example 8

a Show that the vectors (3, –2, 3) and (6, 3, –4) are perpendicular.
b Use the dot product to show that the vectors **a** = (3, 9, –6) and **b** = (–2, –6, 4) are parallel.
c Find the projection of **m** = (2, –4, 5) on **n** = (–1, 3, 4) using the scalar product **m** · **n**.
d Find the projection of **c** = (–1, 3, 2) on **d** = (2, –2, 3) using unit vectors.
e Find the angle between **q** = 3**i** + 4**j** – **k** and **r** = 3**i** – 2**j** – 2**k**.

Solution

a Find the scalar product. $(3, -2, 3) \cdot (6, 3, -4) = 18 - 6 - 12$

Evaluate. $= 0$

Use the fact that cos (90°) = 0 to write the answer. Since the scalar product is zero, the vectors are perpendicular.

b Use the scalar product to calculate cos (θ). $\cos(\theta) = \dfrac{\mathbf{a}\cdot\mathbf{b}}{|\mathbf{a}||\mathbf{b}|}$

Substitute in the known values. $= \dfrac{-6-54-24}{\sqrt{3^2+9^2+(-6)^2}\times\sqrt{(-2)^2+(-6)^2+4^2}}$

Evaluate and simplify. $= \dfrac{-84}{3\sqrt{14}\times2\sqrt{14}}$

$= -1$

Write the interpretation of cos (θ) = –1. The angle is 180°; they are parallel.

c Use the scalar product to calculate cos (θ). $\cos(\theta) = \dfrac{\mathbf{m}\cdot\mathbf{n}}{|\mathbf{m}||\mathbf{n}|}$

Substitute. $= \dfrac{-2-12+20}{\sqrt{2^2+(-4)^2+5^2}\times\sqrt{(-1)^2+3^2+4^2}}$

Evaluate. $= \dfrac{6}{\sqrt{45}\times\sqrt{26}}$

Write the formula for p. $p = |\mathbf{m}|\cos(\theta)$

Substitute in the values. $= \sqrt{45}\times\dfrac{6}{\sqrt{45}\times\sqrt{26}}$

Simplify. $= \dfrac{6}{\sqrt{26}}$

≈ 1.18

Write the answer. The projection of (2, –4, 5) on (–1, 3, 4) is about 1.18 in the direction of **n**.

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d Calculate $\hat{\mathbf{d}}$.

$$\hat{\mathbf{d}} = \frac{1}{|\mathbf{d}|}\mathbf{d}$$

Substitute for the components of **d**.

$$= \frac{1}{\sqrt{2^2+(-2)^2+3^2}}(2, -2, 3)$$

Simplify.

$$= \frac{1}{\sqrt{17}}(2, -2, 3)$$

Write the formula.

$$p = \mathbf{c} \cdot \hat{\mathbf{d}}$$

Substitute.

$$= (-1, 3, 2) \cdot \frac{1}{\sqrt{17}}(2, -2, 3)$$

Rearrange.

$$= \frac{1}{\sqrt{17}}(-2-6+6)$$

Evaluate.

$$= -\frac{2}{\sqrt{17}}$$

$$\approx -0.485$$

Write the answer.

The projection of $(-1, 3, 2)$ on $(2, -2, 3)$ is $-\frac{2}{\sqrt{17}}(2, -2, 3) \approx (-0.97, 0.97, -1.46)$

e Use the scalar product to calculate $\cos(\theta)$.

$$\cos(\theta) = \frac{\mathbf{q} \cdot \mathbf{r}}{|\mathbf{q}||\mathbf{r}|}$$

Substitute in the known values.

$$= \frac{9-8+2}{\sqrt{3^2+4^2+(-1)^2} \times \sqrt{3^2+(-2)^2+(-2)^2}}$$

Evaluate.

$$= \frac{3}{\sqrt{26} \times \sqrt{17}}$$

$$= \frac{3}{\sqrt{442}}$$

Use $\cos^{-1}(\theta)$ to find θ.

$$\theta \approx 81.8°$$

The scalar product is important in the calculation of a number of physical quantities, such as work, energy, electrical potential and electrical power. The work (W) done when a constant force (F) moves an object through a displacement (s) is defined as the component of the force in the direction of motion multiplied by the displacement. This is simply written as $W = \mathbf{F} \cdot \mathbf{s}$.

IMPORTANT

The work done by a constant force **F** that results in a displacement **s** is $\mathbf{F} \cdot \mathbf{s}$.

Example 9

A constant force $\mathbf{a} = 4\mathbf{i} + 3\mathbf{j} + 6\mathbf{k}$ Newtons moves an object from $P(2, -2, 3)$ to $R(5, 4, 0)$. The distance is measured in metres. Find the work done.

Solution

Find the displacement.	$\mathbf{s} = \mathbf{PR} = \mathbf{r} - \mathbf{p}$
Substitute values and simplify.	$= (5, 4, 0) - (2, -2, 3)$ $= (5 - 2, 4 + 2, 0 - 3)$ $= (3, 6, -3)$
Write the formula for work.	$W = \mathbf{F} \cdot \mathbf{s}$
Substitute in the known values.	$= (4\mathbf{i} + 3\mathbf{j} + 6\mathbf{k}) \cdot (3\mathbf{i} + 6\mathbf{j} - 3\mathbf{k})$
Multiply the corresponding components.	$= 12 + 18 - 18 = 12$
State the result.	The work done is 12 Nm.

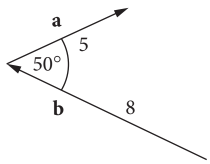
EXERCISE 1.03 The scalar product

Concepts and techniques

WS Scalar product

1 Example 7 The dot product of $2\mathbf{i} - 1\mathbf{j} + 3\mathbf{k}$ and $4\mathbf{i} - 2\mathbf{j} - 1\mathbf{k}$ is:
A 1 B 3 C 7 D 10 E 13

2 Given the two vectors shown on the right, $\mathbf{a} \cdot \mathbf{b}$ is closest to:
A 62.2 B 47.7 C 25.7
D 30.6 E 40



3 Example 8 If $\mathbf{a} = 4\mathbf{i} + 3\mathbf{j} + 5\mathbf{k}$, which of the following is perpendicular to $\mathbf{a}$?
A $-4\mathbf{i} - 3\mathbf{j} - 5\mathbf{k}$ B $5\mathbf{i} - 4\mathbf{k}$ C $5\mathbf{i} + 3\mathbf{j} + 4\mathbf{k}$
D $4\mathbf{j}$ E $4\mathbf{j} - 5\mathbf{k}$

4 If $\mathbf{a} \cdot \mathbf{b} = |\mathbf{a}|^2$, then:
A $\mathbf{a}$ and $\mathbf{b}$ are parallel and equal in magnitude
B $\mathbf{a}$ and $\mathbf{b}$ have equal magnitudes
C $\mathbf{a}$ and $\mathbf{b}$ are perpendicular
D $\mathbf{b}$ is a multiple of $\mathbf{a}$
E None of the above is necessarily true.

5 If $(\mathbf{a} - \mathbf{b}) \cdot (\mathbf{a} + \mathbf{b}) = 0$, then:
A $\mathbf{a}$ and $\mathbf{b}$ are parallel
B $\mathbf{a}$ and $\mathbf{b}$ must be two-dimensional vectors
C $\mathbf{a} = \mathbf{0}$
D $\mathbf{a}$ and $\mathbf{b}$ have the same magnitude
E None of the above is true.

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6 The angle between the vectors $\mathbf{i} + \mathbf{j} + \mathbf{k}$ and $2\mathbf{i} - 2\mathbf{k}$ is closest to:

A 0° B 45° C 90° D 135° E 180°

7 The angle between the vectors $2\mathbf{i} + 3\mathbf{j} - 5\mathbf{k}$ and $-4\mathbf{i} - 6\mathbf{j} + 10\mathbf{k}$ is closest to:

A 0° B 45° C 90° D 180° E 270°

8 In each of the following, the magnitudes of two vectors and the angle between them is given. Find the scalar product in each case.

a 5, 9, 38° b 3, 10, 45° c 11, 7, 90° d 15, 12, 0°
e 7, 2, 30° f 6, 11, 210° g 9, 9, 150° h 8, 12, 300°

9 Find the following scalar products.

a $(9, 120°) \cdot (16, 48°)$ b $(3, 185°) \cdot (16, 200°)$ c $(19, 300°) \cdot (6, 240°)$
d $(3, 160°, 42°) \cdot (5, 160°, 12°)$ e $(9, 280°, 16°) \cdot (4, 280°, -28°)$ f $(15, 45°, -33°) \cdot (8, 45°, 57°)$

10 Find the following scalar products.

a $(3, 8) \cdot (6, -4)$ b $(-2, 10) \cdot (5, 1)$ c $(-6, 8) \cdot (8, -6)$
d $(2, 6, 4) \cdot (5, 10, 1)$ e $(3, -4, 5) \cdot (5, -6, -8)$ f $(9, 2, -12) \cdot (6, -3, 4)$
g $(8, -3, 5) \cdot (-2, -3, -7)$ h $(-4, 2, -8) \cdot (-5, 1, 3)$ i $(1, 0, 5) \cdot (-8, -3, 2)$

11 Find the following scalar products.

a $\begin{bmatrix} 3 \\ -5 \end{bmatrix} \cdot \begin{bmatrix} -5 \\ 3 \end{bmatrix}$ b $\begin{bmatrix} 2 \\ 9 \end{bmatrix} \cdot \begin{bmatrix} -2 \\ -9 \end{bmatrix}$ c $\begin{bmatrix} -3 \\ 8 \end{bmatrix} \cdot \begin{bmatrix} -3 \\ 8 \end{bmatrix}$

d $\begin{bmatrix} 7 \\ -3 \\ 1 \end{bmatrix} \cdot \begin{bmatrix} 4 \\ 2 \\ -5 \end{bmatrix}$ e $\begin{bmatrix} 2 \\ 9 \\ 5 \end{bmatrix} \cdot \begin{bmatrix} 3 \\ -4 \\ 6 \end{bmatrix}$ f $\begin{bmatrix} -1 \\ 0 \\ 7 \end{bmatrix} \cdot \begin{bmatrix} -6 \\ 9 \\ 5 \end{bmatrix}$

12 Given the vectors $\mathbf{a} = 3\mathbf{i} - 4\mathbf{j} + 2\mathbf{k}$, $\mathbf{b} = 2\mathbf{i} + \mathbf{j} - 5\mathbf{k}$, $\mathbf{c} = \mathbf{i} - 5\mathbf{j} - 4\mathbf{k}$, $\mathbf{d} = \mathbf{j} - 2\mathbf{i} + 3\mathbf{k}$ and $\mathbf{e} = 4\mathbf{k} - 3\mathbf{j}$, find:

a $\mathbf{a} \cdot \mathbf{b}$ b $\mathbf{c} \cdot \mathbf{e}$ c $\mathbf{c} \cdot (\mathbf{d} - \mathbf{e})$
d $\mathbf{c} \cdot \mathbf{d} - \mathbf{c} \cdot \mathbf{e}$ e $\mathbf{d} \cdot \mathbf{e}$ f $\mathbf{e} \cdot \mathbf{d}$

13 Find the angle between the following pairs of vectors.

a $(5, -4)$ and $(-2, 1)$ b $(3, 6, -4)$ and $(0, -2, 7)$ c $(0, 3, -2)$ and $(6, 1, 5)$.

14 If $\mathbf{a} = -2\mathbf{i} + 3\mathbf{j} + \mathbf{k}$, $\mathbf{b} = 7\mathbf{i} + 4\mathbf{j} + 5\mathbf{k}$ and $\mathbf{c} = \mathbf{i} - 5\mathbf{j} + 2\mathbf{k}$, find:

a the projection of **b** on **c** b the projection of **c** on **a**
c the projection of **a** on **b** d the projection of **c** on **c**
e the projection of $(\mathbf{a} + \mathbf{c})$ on **b** f the projection of $(\mathbf{a} + \mathbf{b})$ on **c**

15 In each of the following cases, find the angle between **m** and **n**.

a $\mathbf{m} = -4\mathbf{i} + 8\mathbf{j} - 3\mathbf{k}$, $\mathbf{n} = 2\mathbf{i} + \mathbf{j} + \mathbf{k}$ b $\mathbf{m} = \mathbf{i} - 7\mathbf{j} + 4\mathbf{k}$, $\mathbf{n} = 5\mathbf{i} - \mathbf{k}$
c $\mathbf{m} = -2\mathbf{i} - 3\mathbf{j}$, $\mathbf{n} = -6\mathbf{i} + 4\mathbf{k}$ d $\mathbf{m} = 3\mathbf{i} - 5\mathbf{j} - \mathbf{k}$, $\mathbf{n} = 2\mathbf{i} + \mathbf{j} - 3\mathbf{k}$

16 Given the points $A(3, -2, -1)$, $B(1, 5, 4)$, $C(2, 0, -6)$ and $D(-4, 1, 5)$, calculate:

a $\mathbf{AB} \cdot \mathbf{CD}$ b $\mathbf{BD} \cdot \mathbf{CA}$
c the angle between **AB** and **CD** d the angle between **BD** and **CA**
e the projection of **AD** along **BC** f the projection of **AB** along **CD**

17 Example 9 In each of the following, calculate the work done by the force $\mathbf{u}$ in moving an object from P to Q. The force is measured in newtons and the displacement is in metres.

a $\mathbf{u} = -\mathbf{i} + 5\mathbf{j} - 3\mathbf{k}$, $P(4, 0, -7)$ and $Q(2, 4, 0)$

b $\mathbf{u} = 8\mathbf{i} - 4\mathbf{k}$, $P(-1, 2, 5)$ and $Q(4, 1, 0)$

c $\mathbf{u} = 2\mathbf{i} + 3\mathbf{j}$, $P(1, -2, 4)$ and $Q(5, 0, -3)$

18 A force of 12 N causes an object to move 12 m at an angle of 60° to the direction of the force. What is the work done (amount of energy that is transferred)?

19 A constant force of magnitude 5 N in the same direction as the vector $\mathbf{a} = \mathbf{i} + \mathbf{j} + \mathbf{k}$ moves a point from (0, 2, 0) to (0, –3, 0). If the displacement is measured in metres, find the work done.

20 A constant force of magnitude 10 N in the same direction as the positive z-axis moves a point from the origin to (1, 3, 4). If the displacement is measured in metres, find the work done.

Reasoning and communication

21 Find the constant k, such that $\mathbf{a} = 2k\mathbf{i} + 3k\mathbf{j} - 5\mathbf{k}$ is perpendicular to $\mathbf{a} = -2\mathbf{i} + 3\mathbf{j} + 2\mathbf{k}$.

22 A line passes through the points (1, 3, –4) and (2, –5, –2).

a Find a displacement vector parallel to the line.

b Find the cosines of the angles between this vector and each of the axes (the *direction cosines* of the line).

c Find the *direction angles* α, β and γ of the line—the angles between the line and each of the axes.

23 Use a displacement vector and the scalar product to show that, in general,

$$\cos^2(\alpha) + \cos^2(\beta) + \cos^2(\gamma) = 1$$

for the direction cosines of any line in three-dimensional space.

24 Consider the scalar product of unit vectors in two dimensions with directions α and β. Use the scalar product to find an expression for the cosine of the angle between the vectors, $\cos(\alpha - \beta)$.

25 Two forces of 30 N and 50 N act on an object. The 50 N force is at an angle of 35° to the 30 N force. The object moves 8 m at an angle of 15° to the 30 N force, away from the 50 N force.

a Find the total acting force.

b Find the energy transferred (work done) by the:

i total force ii 30 N force iii 50 N force.

c Compare the sizes of the answers to part **b** and interpret this information mathematically.

26 For any vector, prove that $\mathbf{a} \cdot \mathbf{a} = |\mathbf{a}|^2$.

27 Is the angle between the vectors (9, 42°, 31°) and (8, 12°, 31°) equal to, less than or greater than 30°? Explain your reasoning and prove your answer.

1.04 THE VECTOR PRODUCT

You have already extended multiplication by a scalar and the scalar product from two dimensions to three dimensions. There is another kind of product of vectors that can be performed in three dimensions. It is called the **vector product** because it gives a new vector. The new vector is *perpendicular* to the original vectors.

IMPORTANT

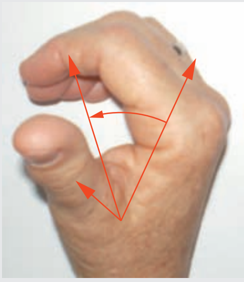
Vector product

The vector product of two vectors $\mathbf{v}_1$ and $\mathbf{v}_2$ with an angle θ between them is defined as

$$\mathbf{v} = \mathbf{v}_1 \times \mathbf{v}_2, \text{ where } |\mathbf{v}| = |\mathbf{v}_1||\mathbf{v}_2| \sin(\theta)$$

and $\mathbf{v}$ is perpendicular to both $\mathbf{v}_1$ and $\mathbf{v}_2$.

The direction of $\mathbf{v}$ is such that $\mathbf{v}_1$, $\mathbf{v}_2$ and $\mathbf{v}$ form a right-handed system.



The vector product is also called the **outer product** or **cross product**.

The vector product may be calculated from vectors in component form using

$$(a_1, a_2, a_3) \times (b_1, b_2, b_3) = (a_2b_3 - a_3b_2, a_3b_1 - a_1b_3, a_1b_2 - a_2b_1)$$

$\mathbf{k}$ and $\mathbf{k}$ are parallel, so $\mathbf{k} \times \mathbf{k} = |\mathbf{k}| \times |\mathbf{k}| \times \sin(0°) = 1 \times 1 \times 0 = 0$.

$\mathbf{i}$ and $\mathbf{j}$ are perpendicular, so $|\mathbf{i} \times \mathbf{j}| = |\mathbf{i}| \times |\mathbf{j}| \times \sin(90°) = 1 \times 1 \times 1 = 1$.

$\mathbf{i}$, $\mathbf{j}$ and $\mathbf{k}$ make a right-handed system, so $\mathbf{i} \times \mathbf{j} = \mathbf{k}$.

If you consider all the possible orders, you get $\mathbf{j} \times \mathbf{k} = \mathbf{i}$ and $\mathbf{k} \times \mathbf{i} = \mathbf{i}$.

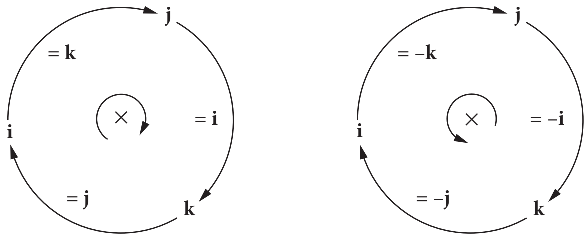
The reverse orders give negatives: $\mathbf{j} \times \mathbf{i} = -\mathbf{k}$, $\mathbf{k} \times \mathbf{j} = -\mathbf{i}$ and $\mathbf{i} \times \mathbf{k} = -\mathbf{j}$.

You can set out all the possible products of $a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k}$ and $b_1\mathbf{i} + b_2\mathbf{j} + b_3\mathbf{k}$ in a square.

	$b_1\mathbf{i}$	$b_2\mathbf{j}$	$b_3\mathbf{k}$
$a_1\mathbf{i}$	$a_1\mathbf{i} \times b_1\mathbf{i}$	$a_1\mathbf{i} \times b_2\mathbf{j}$	$a_1\mathbf{i} \times b_3\mathbf{k}$
$a_2\mathbf{j}$	$a_2\mathbf{j} \times b_1\mathbf{i}$	$a_2\mathbf{j} \times b_2\mathbf{j}$	$a_2\mathbf{j} \times b_3\mathbf{k}$
$a_3\mathbf{k}$	$a_3\mathbf{k} \times b_1\mathbf{i}$	$a_3\mathbf{k} \times b_2\mathbf{j}$	$a_3\mathbf{k} \times b_3\mathbf{k}$

The products that are not zero are the ones on the outside.

One way to remember which are positive and which are negative is to imagine the letters $\mathbf{i} \rightarrow \mathbf{j} \rightarrow \mathbf{k}$ as a clockwise cycle: if you multiply in order of the cycle, the result is positive; if you multiply in the opposite order, the result is negative. This is shown as a diagram below.



Some students and teachers prefer the '**determinant-like**' method.

To find the vector product of $\mathbf{a} = (a_1, a_2, a_3)$ and $\mathbf{b} = (b_1, b_2, b_3)$, write the unit vectors **i**, **j**, **k** and the components in a square pattern as shown.

$$\left\{\begin{matrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{matrix}\right\}$$

To find each component of $\mathbf{a} \times \mathbf{b}$, cross-multiply and subtract the components in the square *outside* the column for that component, making the **j** *component negative*.

$$\left\{\begin{matrix} \mathbf{i} & & \\ & a_2 & a_3 \\ & b_2 & b_3 \end{matrix}\right\} \quad \left\{\begin{matrix} & \mathbf{j} & \\ a_1 & & a_3 \\ b_1 & & b_3 \end{matrix}\right\} \quad \left\{\begin{matrix} & & \mathbf{k} \\ a_1 & a_2 & \\ b_1 & b_2 & \end{matrix}\right\}$$

$\mathbf{a} \times \mathbf{b} = (a_2b_3 - a_3b_2)\mathbf{i} - (a_1b_3 - a_3b_1)\mathbf{j} + (a_1b_2 - a_2b_1)\mathbf{k}$

This method is a little more tedious to write out, but has the advantage of a clear pattern.

Example 10

a Find the vector product between two vectors of magnitudes 13 and 18 if the angle between them is 48°.

b Find the vector product of $\mathbf{u} = (1, -2, 4)$ and $\mathbf{v} = (-2, 3, -1)$.

Solution

a	Find the magnitude.	$\lvert\mathbf{v}\rvert = \lvert\mathbf{v}_1\rvert\,\lvert\mathbf{v}_2\rvert \sin(\theta)$ $= 13 \times 18 \times \sin(48°)$ ≈ 173.9
	Write the answer.	The vector product is about 174 in a direction perpendicular to the plane of the original vectors.

b Write in terms of **i**, **j** and **k**.

$\mathbf{u} = \mathbf{i} - 2\mathbf{j} + 4\mathbf{k}$, $\mathbf{v} = -2\mathbf{i} + 3\mathbf{j} - \mathbf{k}$

Use the cycle to get signs.

$$\begin{aligned}\mathbf{u} \times \mathbf{v} = {} & [-2 \times (-1) - 4 \times 3]\mathbf{i} \\ & + [4 \times (-2) - 1 \times (-1)]\mathbf{j} \\ & + [1 \times 3 - (-2) \times (-2)]\mathbf{k}\end{aligned}$$

Simplify.

$$= -10\mathbf{i} - 7\mathbf{j} - \mathbf{k}$$

You can use the determinant-like method instead. Write in the square form.

$$\left\{\begin{matrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & -2 & 4 \\ -2 & 3 & -1 \end{matrix}\right\}$$

One, two, miss a few!

$$\left| \mathbf{i} \begin{matrix} -2 & 4 \\ 3 & -1 \end{matrix} \right| \left| \mathbf{j} \begin{matrix} 1 & 4 \\ -2 & -1 \end{matrix} \right| \left| \mathbf{k} \begin{matrix} 1 & -2 \\ -2 & 3 \end{matrix} \right|$$

Write out, with the **j** component negative.

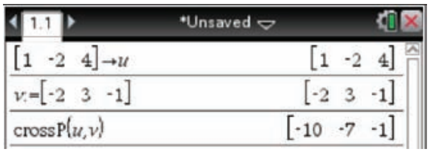
$$\begin{aligned}\mathbf{u} \times \mathbf{v} = {} & [-2 \times (-1) - 4 \times 3]\mathbf{i} \\ & - [1 \times (-1) - 4 \times (-2)]\mathbf{j} \\ & + [1 \times 3 - (-2) \times (-2)]\mathbf{k}\end{aligned}$$

Simplify.

$$= -10\mathbf{i} - 7\mathbf{j} - \mathbf{k}$$

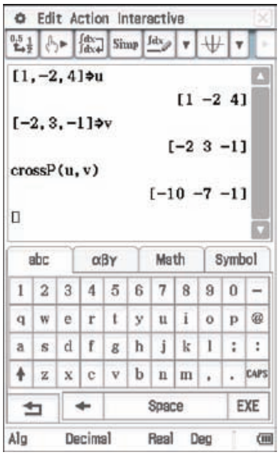
TI-Nspire CAS

Enter the vectors.
Type crossP or use [menu], 7: Matrix & Vector, C: Vector and 2: Cross Product.



ClassPad

Enter the vectors. To obtain the cross product command, tap **Action Vector** then **crossP**. Enter u [,] v [)] [EXE].



Write the answer in the original form.

The vector product is (−10, −7, −1).

INVESTIGATION Properties of the vector product

1. Consider two vectors $v_1 = (a_1, a_2, a_3)$ and $v_2 = (b_1, b_2, b_3)$.

   Use the scalar product to find the cosine of the angle between them.

   Use the result that $\sin(\theta) = \sqrt{1 - \cos^2(\theta)}$ to find the sine of the angle between them.

   Hence find the magnitude of the vector product.

2. Consider the vector product that is obtained when using the component rule method. Find the norm of the vector product.

   Compare your answer with the size obtained from the scalar product method.

3. Now use the scalar product to show that the component form of the vector product is perpendicular to both of the original vectors.

4. Comment on the difficulties of proving that the component form of the vector product is the same as the form that uses the angle between the vectors.

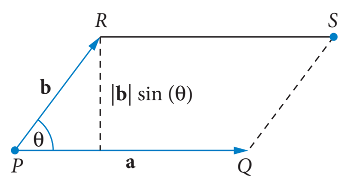
The properties of the vector product are summarised below.

IMPORTANT

Properties of the vector product

1. $|\mathbf{a} \times \mathbf{b}| = |\mathbf{a}|\,|\mathbf{b}| \sin(\theta)$, where θ is the angle between **a** and **b**.
2. The vector product $\mathbf{a} \times \mathbf{b}$ is perpendicular to both **a** and **b**.
3. $\mathbf{a} \times \mathbf{a} = \mathbf{0}$ for any vector **a**.
4. The vector product is left- and right-distributive over vector addition.
5. $\mathbf{a} \times \mathbf{b} = \mathbf{0} \Rightarrow \mathbf{a} = \mathbf{0}$ or $\mathbf{b} = \mathbf{0}$ or **a** and **b** are parallel.
6. $\mathbf{a} \times \mathbf{b} = -\mathbf{b} \times \mathbf{a}\ \forall$ vectors **a**, **b**.
7. $m(\mathbf{a} \times \mathbf{b}) = (m\mathbf{a}) \times \mathbf{b} = \mathbf{a} \times (m\mathbf{b})\ \forall\ m \in \boldsymbol{R}$ and vectors **a**, **b**.

You can interpret $|\mathbf{a} \times \mathbf{b}|$ geometrically. Consider two vectors $\mathbf{a} = \mathbf{PR}$ and $\mathbf{b} = \mathbf{PQ}$ with an angle of θ between them as shown here. The parallelogram formed with adjacent sides PR and QS is $PQRS$.



The height of $PQRS$ is $|\mathbf{b}| \sin(\theta)$.

Hence, the area of $PQRS$ is $|\mathbf{a}|\,|\mathbf{b}| \sin(\theta)$.

But $|\mathbf{a} \times \mathbf{b}| = |\mathbf{a}|\,|\mathbf{b}| \sin(\theta)$

IMPORTANT

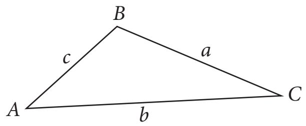
The magnitude of the vector product $\mathbf{a} \times \mathbf{b}$ equals the area of the parallelogram determined by **a** and **b**.

Example 11

Use the vector product to find the area of the triangle with vertices $A(-1, 3, 2)$, $B(3, 5, 1)$ and $C(1, 6, -2)$.

Solution

Make a general sketch of the triangle. Label the sides and angles.



Write a formula for the area.

$$\text{Area} = \frac{1}{2}ab\sin(C)$$

Express it in terms of the vector product.

$$= \frac{1}{2}|\mathbf{BC} \times \mathbf{AC}|$$

Express the sides in terms of position vectors.

$$= \frac{1}{2}|(\mathbf{c} - \mathbf{b}) \times (\mathbf{c} - \mathbf{a})|$$

Find the sides.

$$\begin{aligned}\mathbf{c} - \mathbf{b} &= (1, 6, -2) - (3, 5, 1)\\ &= (-2, 1, -3)\\ &= -2\mathbf{i} + \mathbf{j} - 3\mathbf{k}\\ \mathbf{c} - \mathbf{a} &= (1, 6, -2) - (-1, 3, 2)\\ &= (2, 3, -4)\\ &= 2\mathbf{i} + 3\mathbf{j} - 4\mathbf{k}\end{aligned}$$

Find the vector product.

$$\begin{aligned}(\mathbf{c} - \mathbf{b}) \times (\mathbf{c} - \mathbf{a}) &= (-2\mathbf{i} + \mathbf{j} - 3\mathbf{k}) \times (2\mathbf{i} + 3\mathbf{j} - 4\mathbf{k})\\ &= [1 \times (-4) - (-3) \times 3]\mathbf{i}\\ &\quad + [(-3) \times 2 - (-2) \times (-4)]\mathbf{j}\\ &\quad + [-2 \times 3 - 1 \times 2]\mathbf{k}\\ &= 5\mathbf{i} - 14\mathbf{j} - 8\mathbf{k}\end{aligned}$$

Find the norm.

$$\begin{aligned}|(\mathbf{c} - \mathbf{b}) \times (\mathbf{c} - \mathbf{a})| &= \sqrt{5^2 + (-14)^2 + (-8)^2}\\ &= \sqrt{285}\end{aligned}$$

Find the area.

$$\begin{aligned}\text{Area} &= \frac{1}{2}|(\mathbf{c} - \mathbf{b}) \times (\mathbf{c} - \mathbf{a})|\\ &= \frac{1}{2} \times \sqrt{285}\\ &\approx 8.44\end{aligned}$$

Write the answer.

The area of the triangle is about 8.44 square units.

Example 12

Find a vector normal to the plane containing the points $A(3, -2, 1)$, $B(1, 3, 5)$ and $C(-2, 4, -1)$.

Solution

The vector product of two vectors is normal to the plane containing the vectors. You can use the product $\mathbf{AB} \times \mathbf{AC}$, say, to find a normal vector.

Find two vectors in the plane.

$$\begin{aligned}\mathbf{AB} &= \mathbf{b} - \mathbf{a}\\ &= (\mathbf{i} + 3\mathbf{j} + 5\mathbf{k}) - (3\mathbf{i} - 2\mathbf{j} + \mathbf{k})\\ &= -2\mathbf{i} + 5\mathbf{j} + 4\mathbf{k}\\ \mathbf{AC} &= \mathbf{c} - \mathbf{a}\\ &= (-2\mathbf{i} + 4\mathbf{j} - \mathbf{k}) - (3\mathbf{i} - 2\mathbf{j} + \mathbf{k})\\ &= -5\mathbf{i} + 6\mathbf{j} - 2\mathbf{k}\end{aligned}$$

Find the vector product.

$$\begin{aligned}\mathbf{AB} \times \mathbf{AC} &= (-2\mathbf{i} + 5\mathbf{j} + 4\mathbf{k}) \times (-5\mathbf{i} + 6\mathbf{j} - 2\mathbf{k})\\ &= [5 \times (-2) - 4 \times 6]\mathbf{i}\\ &\quad - [4 \times (-5) - (-2) \times (-2)]\mathbf{j}\\ &\quad + [-2 \times 6 - 5 \times (-5)]\mathbf{k}\\ &= -34\mathbf{i} + 24\mathbf{j} + 13\mathbf{k}\end{aligned}$$

Write the answer in the original form.

The vector $(-34, 24, 13)$ is normal to the plane.

In the last two examples, you could choose different displacement vectors for the vector product. The only difference in the end result would be a possible change of sign, since the area of the triangle is unchanged.

EXERCISE 1.04 The vector product

Concepts and techniques

WS Vector product

1 Example 10 Given $\mathbf{a} = \mathbf{i} + 2\mathbf{j} + 3\mathbf{k}$, $\mathbf{b} = 3\mathbf{i} + 2\mathbf{j} + \mathbf{k}$ and $\mathbf{c} = -\mathbf{i} - 2\mathbf{j} - 3\mathbf{k}$, $\mathbf{a} \times \mathbf{b}$ is equal to:

A $\mathbf{a} \cdot \mathbf{b}$ B $\mathbf{b} \times \mathbf{c}$ C $\mathbf{b} \times \mathbf{a}$ D $\mathbf{b} \cdot \mathbf{a}$ E $\mathbf{c} \times \mathbf{b}$

2 Given $P(6, -3, 9)$ and $Q(-4, 2, -1)$, $\mathbf{p} \times \mathbf{q}$ is equal to:

A $\mathbf{OP} \times \mathbf{PQ}$ B $\mathbf{PQ} \times \mathbf{p}$ C $\mathbf{PQ} \times \mathbf{q}$ D $\mathbf{0}$ E $\mathbf{PQ} \times \mathbf{OQ}$

3 Given the vectors $\mathbf{m}$ and $\mathbf{n}$, $\mathbf{m} \times \mathbf{n}$ is:

A parallel to **m** and perpendicular to **n**
B parallel to **m** and **n**
C parallel to **n** and perpendicular to **m**
D perpendicular to **m** and **n**
E None of the above is true.

4 In each of the following cases, the magnitude of two vectors and the angle between them is given. Find the vector products of the pairs of vectors in each case.

a 3, 8, 120° b 10, 2, 78° c 12, 12, 200° d 5, 2, 0°
e 6, 8, 47° f 5, 14, 330° g 7, 11, 90° h 4, 7, 60°

5 Find the following vector products.

a $(3, 180°) \times (79, 25°)$

b $(6, 251°) \times (32, 80°)$

c $(73, 60°) \times (9, -20°)$

d $(6, 79°, 28°) \times (1, 79°, 78°)$

e $(3, 150°, 79°) \times (2, 150°, -85°)$

f $(7, 21°, -66°) \times (5, 21°, 14°)$

6 Find the following vector products.

a $(6, 5) \times (9, -2)$

b $(-8, 70) \times (1, 7)$

c $(-9, 5) \times (5, -9)$

d $(8, 9, 2) \times (1, 7, 7)$

e $(6, -2, 1) \times (1, -9, -5)$

f $(3, 8, -17) \times (9, -6, 2)$

g $(5, -6, 1) \times (-8, -6, -4)$

h $(-2, 8, -5) \times (-1, 7, 6)$

i $(7, 0, 1) \times (-5, -6, 8)$

7 Find the following vector products.

a $\begin{bmatrix} 3 \\ -5 \end{bmatrix} \times \begin{bmatrix} -5 \\ 3 \end{bmatrix}$

b $\begin{bmatrix} 2 \\ 9 \end{bmatrix} \times \begin{bmatrix} -2 \\ -9 \end{bmatrix}$

c $\begin{bmatrix} -3 \\ 8 \end{bmatrix} \times \begin{bmatrix} -3 \\ 8 \end{bmatrix}$

d $\begin{bmatrix} 7 \\ -3 \\ 1 \end{bmatrix} \times \begin{bmatrix} 4 \\ 2 \\ -5 \end{bmatrix}$

e $\begin{bmatrix} 2 \\ 9 \\ 5 \end{bmatrix} \times \begin{bmatrix} 3 \\ -4 \\ 6 \end{bmatrix}$

f $\begin{bmatrix} 1 \\ 0 \\ 7 \end{bmatrix} \times \begin{bmatrix} -6 \\ 9 \\ 5 \end{bmatrix}$

8 Given the vectors $\mathbf{a} = 3\mathbf{i} - 4\mathbf{j} + 2\mathbf{k}$, $\mathbf{b} = 2\mathbf{i} + \mathbf{j} - 5\mathbf{k}$, $\mathbf{c} = \mathbf{i} - 5\mathbf{j} - 4\mathbf{k}$, $\mathbf{d} = \mathbf{j} - 2\mathbf{i} + 3\mathbf{k}$ and $\mathbf{e} = 4\mathbf{k} - 3\mathbf{j}$, find:

a $\mathbf{a} \times \mathbf{b}$

b $\mathbf{c} \times \mathbf{e}$

c $\mathbf{c} \times (\mathbf{d} - \mathbf{e})$

d $\mathbf{c} \times \mathbf{d} - \mathbf{c} \times \mathbf{e}$

e $\mathbf{e} \times \mathbf{d}$

f $\mathbf{d} \times \mathbf{e}$

g $\mathbf{a} \times (\mathbf{b} \times \mathbf{c})$

h $(\mathbf{a} \times \mathbf{b}) \times \mathbf{c}$

9 Given the vectors $\mathbf{a} = -\mathbf{i} + 6\mathbf{j} - 4\mathbf{k}$, $\mathbf{b} = 7\mathbf{i} - 4\mathbf{j} + 11\mathbf{k}$, $\mathbf{c} = -2\mathbf{i} + 12\mathbf{j} + 9\mathbf{k}$, $\mathbf{d} = 14\mathbf{i} - 7\mathbf{j}$, $\mathbf{e} = 12\mathbf{i} + 28\mathbf{j} + 15\mathbf{k}$, $\mathbf{f} = 12\mathbf{i} - 8\mathbf{j} - 2\mathbf{k}$, $\mathbf{g} = -3\mathbf{i} - 2\mathbf{j} - \mathbf{k}$ and $\mathbf{h} = 34\mathbf{i} - 22\mathbf{j} - 24\mathbf{k}$, find

a $\mathbf{a} \times \mathbf{b}$

b $\mathbf{c} \times \mathbf{a}$

c $\mathbf{a} \times \mathbf{c}$

d $\mathbf{c} \times \mathbf{d}$

e $\mathbf{b} \times (\mathbf{c} \times \mathbf{d})$

f $\mathbf{b} \times \mathbf{c}$

g $(\mathbf{b} \times \mathbf{c}) \times \mathbf{d}$

h $\mathbf{e} \times \mathbf{f}$

i $\mathbf{e} \times (\mathbf{g} + \mathbf{h})$

j $\mathbf{e} \times \mathbf{g} + \mathbf{e} \times \mathbf{h}$

k $\mathbf{e} \times \mathbf{g} + \mathbf{h} \times \mathbf{e}$

10 Example 11 If a triangle has vertices at $O(0, 0, 0)$, $P(1, 2, 3)$ and $Q(-1, 2, -3)$, then the area of $\triangle OPQ$ is:

A $\sqrt{13}$

B $2\sqrt{10}$

C $2\sqrt{13}$

D 10

E $4\sqrt{10}$

11 Use the vector product to find the areas of triangles with vertices:

a $(2, 3)$, $(6, 4)$ and $(4, -2)$

b $(3, 2, 5)$, $(6, -3, 4)$ and $(-2, -3, 1)$

c $(2, 3, 4)$, $(-1, 2, 0)$ and $(3, 0, 2)$

12 Use the vector product to find the areas of parallelograms with vertices:

a $(3, -2)$, $(5, 3)$, $(7, 5)$ and $(5, 0)$

b $(4, -2, 3)$, $(-1, 3, 4)$, $(2, 4, -2)$ and $(7, -1, -3)$.

13 Example 12 Find a vector normal to the plane containing the points:

a $(3, 2, 0)$, $(1, 6, -2)$ and $(7, 4, 5)$

b $(1, 8, 7)$, $(-2, -1, 3)$ and $(1, -2, -5)$.

Reasoning and communication

14 If $\mathbf{a} = 5\mathbf{i} + 7\mathbf{j} + 2\mathbf{k}$, $\mathbf{b} = 3\mathbf{i} + 3\mathbf{j} + 4\mathbf{k}$ and $\mathbf{c} = 2\mathbf{i} + \mathbf{j} + 5\mathbf{k}$, show that:

$\mathbf{a} \times \mathbf{b} + \mathbf{b} \times \mathbf{c} + \mathbf{c} \times \mathbf{a} = \mathbf{0}$

15 For two vectors **a** and **b**, $\mathbf{a} \times \mathbf{b} = \mathbf{b} \times \mathbf{a}$. Write a relationship between **a** and **b** and explain your reasoning.

16 Let $A(1, 3, 4)$, $B(2, -2, 6)$ and $C(5, 2, 3)$ be three vertices of the parallelogram $ABCD$. Find:
  a the coordinates of D
  b the area of $ABCD$

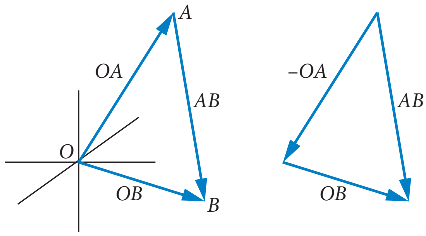
17 Find a unit vector that is normal to the plane containing the points $(1, 2, -3)$, $(-2, 4, 1)$ and $(5, -1, -3)$.

18 Use a counterexample to show that the vector product is not commutative.

19 Use a counterexample to show that the vector product is not associative.

1.05 CARTESIAN GEOMETRY

You should remember the following, which are important for this section.



You can write a vector as a linear combination of vectors in particular directions. This is called **resolution** of the vector. **Orthogonal** components are at right angles to each other.

A displacement is a change of position. The displacement **AB** is the change of position from point A to point B.

The position vector of a point is its displacement from the origin. The position vector of point A is written as **OA**.

Point O is always used for the origin, and the symbol **a** may be used instead of **OA** for the position vector of point A, so $\mathbf{AB} = \mathbf{OB} - \mathbf{OA} = \mathbf{b} - \mathbf{a}$.

The scalar product can be used for geometric proofs involving perpendicularity. The following results are useful in geometric proofs.

- $k\mathbf{a}$, where $k \in \boldsymbol{R}^+$, is a vector in the same direction as **a** but with a magnitude of $k|\mathbf{a}|$.
- $-k\mathbf{a}$, where $k \in \boldsymbol{R}^+$, is a vector in the opposite direction to **a** but with a magnitude of $k|\mathbf{a}|$.
- If **a** is parallel to **b**, there exists some real number k ($k \neq 0$) such that $\mathbf{b} = k\mathbf{a}$. Conversely, if $\mathbf{b} = k\mathbf{a}$, then **a** and **b** are parallel.
- If **AB** and **BC** are parallel, then A, B and C are collinear.
- If $\mathbf{a} \cdot \mathbf{b} = 0$, then $\mathbf{a} = \mathbf{0}$ or $\mathbf{b} = \mathbf{0}$ or **a** and **b** are perpendicular.

  Or if **a** and **b** are perpendicular, $\mathbf{a} \cdot \mathbf{b} = 0$.
- $\mathbf{a} \cdot \mathbf{a} = |\mathbf{a}|^2$

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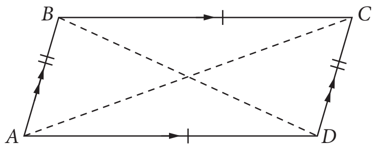
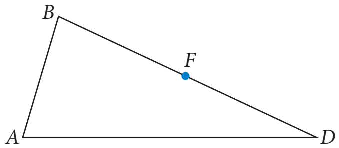
Example 13

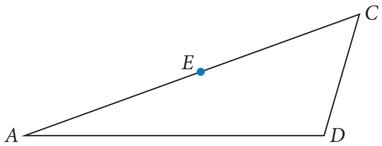
Show that the diagonals of a parallelogram bisect each other.

Solution

Draw a diagram of a parallelogram *ABCD* and diagonals *BD* and *AC*.
Show the midpoints *F* and *E* of *BD* and *AC* respectively.







Stating that the diagonals bisect each other is the same as saying that the midpoints of *BD* and *AC* coincide. Try to prove that *E* and *F* are the same point.

Find the position vector of *E*. $\mathbf{OE} = \mathbf{OA} + \frac{1}{2}\mathbf{AC}$

Simplify.

$$= \mathbf{a} + \frac{1}{2}(\mathbf{c} - \mathbf{a})$$
$$= \mathbf{a} + \frac{1}{2}\mathbf{c} - \frac{1}{2}\mathbf{a}$$
$$= \frac{1}{2}\mathbf{a} + \frac{1}{2}\mathbf{c}$$
$$= \frac{1}{2}(\mathbf{a} + \mathbf{c})$$

Find the position vector of *F*. $\mathbf{OF} = \mathbf{OB} + \frac{1}{2}\mathbf{BD}$

Simplify as above. $= \frac{1}{2}(\mathbf{b} + \mathbf{d})$

Use parallelogram properties. $\mathbf{AB} = \mathbf{DC}$, so $\mathbf{b} - \mathbf{a} = \mathbf{c} - \mathbf{d}$

Rearrange to suit **OE** and **OF**. $\mathbf{b} + \mathbf{d} = \mathbf{a} + \mathbf{c}$

Substitute in the **OF** expression.

$$\mathbf{OF} = \frac{1}{2}(\mathbf{b} + \mathbf{d})$$
$$= \frac{1}{2}(\mathbf{a} + \mathbf{c})$$

Use the **OE** expression. $= \mathbf{OE}$

Write a conclusion. Since **OE** and **OF** are the same, *E* and *F* are the same, so the diagonals of the parallelogram bisect each other.

You should remember the following important result demonstrated in Example **13**.

IMPORTANT

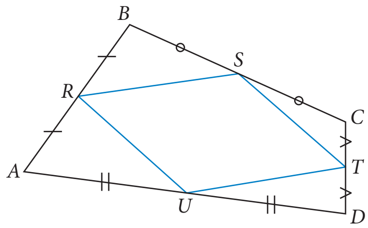
If *P* is the **midpoint** of *AB*, its position vector is given by $\mathbf{p} = \frac{1}{2}(\mathbf{a} + \mathbf{b})$.

Example 14

Show that the midpoints of a quadrilateral form a parallelogram.

Solution

Let $ABCD$ be a quadrilateral, and let the midpoints of AB, BC, CD and DA be R, S, T and U respectively.
Draw a diagram.



Use the fact that R, S, T and U are midpoints.

$\mathbf{AB} = 2\mathbf{AR} = 2\mathbf{RB}$
$\mathbf{BC} = 2\mathbf{BS} = 2\mathbf{SC}$
$\mathbf{CD} = 2\mathbf{CT} = 2\mathbf{TD}$
$\mathbf{DA} = 2\mathbf{DU} = 2\mathbf{UA}$

The displacement from A to A is zero.

$\mathbf{AB} + \mathbf{BC} + \mathbf{CD} + \mathbf{DA} = \mathbf{0}$

Substitute.

$2\mathbf{RB} + 2\mathbf{BS} + 2\mathbf{TD} + 2\mathbf{DU} = \mathbf{0}$
$\mathbf{RB} + \mathbf{BS} + \mathbf{TD} + \mathbf{DU} = \mathbf{0}$

Rearrange.

$\mathbf{RB} + \mathbf{BS} = -(\mathbf{TD} + \mathbf{DU})$

Use vector addition.

$\mathbf{RS} = -\mathbf{TU} = \mathbf{UT}$

Write a conclusion.

The side RS is equal and parallel to the side UT. This is sufficient to show that $RSTU$ is a parallelogram by itself, but you could also show that ST is equal and parallel to RU by choosing the other halves of the sides.

Note that in the previous proof you could use the fact that the sum of the displacement vectors of any closed shape must be the zero vector. It is also useful to remember that the sum of a series of displacements 'end-to-end' can be replaced by the total displacement by using just the first and last points, as in $\mathbf{AB} + \mathbf{BP} + \mathbf{PQ} + \mathbf{QE} = \mathbf{AE}$.

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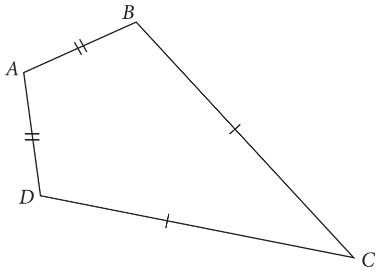
Example 15

Show that the diagonals of a kite are perpendicular.

Solution

Draw a kite $ABCD$ such that AC is a diagonal. The short sides are of equal length and the long sides are of equal length.



Use equality of sides.

$$|\mathbf{AB}| = |\mathbf{AD}| \text{ and } |\mathbf{BC}| = |\mathbf{DC}|$$

Square.

$$|\mathbf{AB}|^2 = |\mathbf{AD}|^2 \text{ and } |\mathbf{BC}|^2 = |\mathbf{DC}|^2$$

Use $|\mathbf{v}|^2 = \mathbf{v} \cdot \mathbf{v}$.

$$\mathbf{AB} \cdot \mathbf{AB} = \mathbf{AD} \cdot \mathbf{AD}$$

Substitute position vectors.

$$(\mathbf{b} - \mathbf{a}) \cdot (\mathbf{b} - \mathbf{a}) = (\mathbf{d} - \mathbf{a}) \cdot (\mathbf{d} - \mathbf{a})$$

Expand the brackets.

$$\mathbf{b}^2 - 2\mathbf{a} \cdot \mathbf{b} + \mathbf{a}^2 = \mathbf{d}^2 - 2\mathbf{a} \cdot \mathbf{d} + \mathbf{a}^2$$

Rearrange.

$$2\mathbf{a} \cdot \mathbf{d} - 2\mathbf{a} \cdot \mathbf{b} = \mathbf{d}^2 - \mathbf{b}^2$$

Simplify and rearrange.

$$\mathbf{a} \cdot (\mathbf{d} - \mathbf{b}) = \frac{1}{2}(\mathbf{d}^2 - \mathbf{b}^2)$$

Do the same for the other sides.

$$\begin{aligned}\mathbf{BC} \cdot \mathbf{BC} &= \mathbf{DC} \cdot \mathbf{DC}\\ (\mathbf{c} - \mathbf{b}) \cdot (\mathbf{c} - \mathbf{b}) &= (\mathbf{c} - \mathbf{d}) \cdot (\mathbf{c} - \mathbf{d})\\ \mathbf{c}^2 - 2\mathbf{b} \cdot \mathbf{c} + \mathbf{b}^2 &= \mathbf{c}^2 - 2\mathbf{c} \cdot \mathbf{d} + \mathbf{d}^2\\ \mathbf{c} \cdot (\mathbf{d} - \mathbf{b}) &= \frac{1}{2}(\mathbf{d}^2 - \mathbf{b}^2)\end{aligned}$$

Find the dot products of the diagonals.

$$\begin{aligned}\mathbf{AC} \cdot \mathbf{BD} &= (\mathbf{c} - \mathbf{a}) \cdot (\mathbf{d} - \mathbf{b})\\ &= \mathbf{c} \cdot (\mathbf{d} - \mathbf{b}) - \mathbf{a} \cdot (\mathbf{d} - \mathbf{b})\\ &= \frac{1}{2}(\mathbf{d}^2 - \mathbf{b}^2) - \frac{1}{2}(\mathbf{d}^2 - \mathbf{b}^2)\\ &= 0\end{aligned}$$

Write a conclusion.

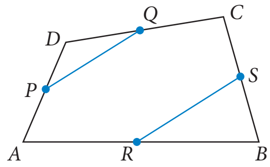
Since $\mathbf{AC} \cdot \mathbf{BD} = 0$, the vectors are perpendicular, so the diagonals of a kite intersect at right angles.

EXERCISE 1.05 Cartesian geometry

Concepts and techniques

WS Geometric proofs using vectors

1 Examples 13–15 $ABCD$ shown on the right is a quadrilateral. Points P, Q, R and S are the midpoints of AD, DC, AB and BC respectively.



Complete the missing parts of the following proof that **PQ** and **RS** are parallel and $|\mathbf{PQ}| = |\mathbf{RS}|$.

Proof

Express **q** in terms of **c** and **d**. $\mathbf{q} = \dfrac{\ldots + \ldots}{2}$

Express **p** in terms of **a** and **d**. $\mathbf{p} = \dfrac{\ldots + \ldots}{2}$

Express **s** in terms of **c** and **b**. $\mathbf{s} = \dfrac{\ldots + \ldots}{2}$

Express **r** in terms of **a** and **b**. $\mathbf{r} = \dfrac{\ldots + \ldots}{2}$

Write **PQ** in terms of **a**, **c** and **d**. $\mathbf{PQ} = \dfrac{\ldots + \ldots}{2} - \dfrac{\ldots + \ldots}{2}$

Simplify. $\mathbf{PQ} = \dfrac{\ldots - \ldots}{2}$

Write **RS** in terms of **a**, **b** and **c**. $\mathbf{RS} = \dfrac{\ldots + \ldots}{2} - \dfrac{\ldots + \ldots}{2}$

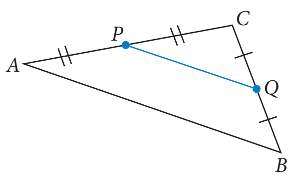
Simplify. $\mathbf{RS} = \dfrac{\ldots - \ldots}{2}$

Hence. $\mathbf{PQ} = \ldots$

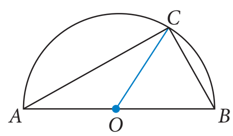
Therefore $\mathbf{PQ} \parallel \ldots$ and $|\mathbf{PQ}| = |\ldots|$

Reasoning and communication

2 For $\triangle ABC$, P and Q are the midpoints of AC and BC respectively. Show that $\mathbf{PQ} \parallel \mathbf{AB}$ and equal to half its length.



3 Given a semicircle with centre O and diameter AB, prove that the angle subtended at the circumference is a right angle.



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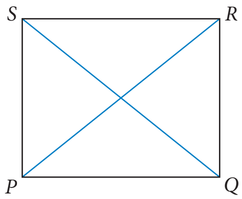
4 $PQRS$ is a rectangle. Use the scalar product to prove that if $\mathbf{PR} \perp \mathbf{SQ}$, then $PQRS$ is a square.



5 Find a vector expression in terms of **a** and **b** for the position vector of the point P on AB that divides it in the ratio $k_1 : k_2$.

6 In the triangle ABC, a median is drawn from A to the midpoint D of the side BC. The point M divides AD in the ratio 2 : 1.

a Find an expression for **m** in terms of **a**, **b** and **c**.

b Use the result from part **a** to show that the medians are concurrent.

c Write an expression for the position vector of the centroid of ABC.

7 In a triangle ABC, the point D divides BC in the ratio $|\mathbf{AB}| : |\mathbf{AC}|$.

a Express **d** in terms of **a**, **b** and **c**.

b Use the scalar product to find $\cos(\angle DAC) - \cos(\angle DAB)$.

c Use your results to make an inference about the angle bisectors of a triangle.

8 If G is the centroid of a triangle ABC, prove that $\mathbf{GA} + \mathbf{GB} + \mathbf{GC} = \mathbf{0}$.

9 $ABCD$ is a tetrahedron. H is the point that divides the line joining A to the centroid of BCD in the ratio 3 : 1.

a Find an expression for the position vector of H in terms of **a**, **b**, **c** and **d**.

b Show that the four lines joining the vertices of $ABCD$ to the centroids of the opposite faces are concurrent.

10 Given a tetrahedron $ABCD$, find an expression for the midpoint of the line joining the midpoints of the edges AB and CD.

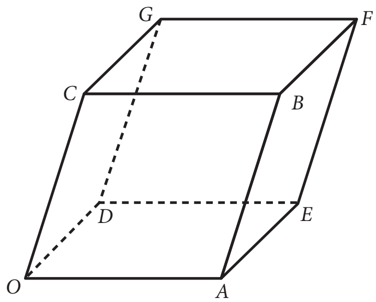
11 Use the scalar product to prove that opposite angles in a parallelogram are equal.

12 Use the scalar product to prove that, if the opposite sides of a quadrilateral are parallel and equal in length, the quadrilateral must be a parallelogram.

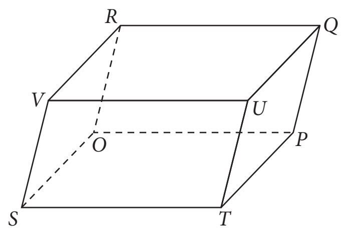
13 Use the vector product to prove that a quadrilateral whose diagonals bisect each other and are equal in length must be a rectangle.

14 Prove that one diagonal of a kite is bisected by the other.

15 Consider the parallelepiped $OABCDEFG$. If $\mathbf{OA} = \mathbf{a}$, $\mathbf{OD} = \mathbf{d}$ and $\mathbf{OC} = \mathbf{c}$, show that
$|\mathbf{OF}|^2 = |\mathbf{a}|^2 + |\mathbf{c}|^2 + |\mathbf{d}|^2 + 2(\mathbf{d} \cdot \mathbf{a} + \mathbf{d} \cdot \mathbf{c} + \mathbf{a} \cdot \mathbf{c})$

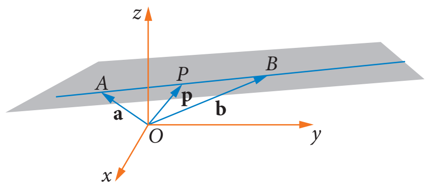


16 Consider the parallelepiped $OPQRSTUV$. Show that the diagonals SQ and RT bisect each other.



1.06 LINES IN 3D

Consider the line AB shown here in three-dimensional space.



A and B are points on the line, and have position vectors $\mathbf{a}$ and $\mathbf{b}$ respectively. P is any other point on the line, and has position vector $\mathbf{p}$. Because A, P and B are collinear, $\mathbf{AB}$ and $\mathbf{AP}$ are parallel, so $\mathbf{AP}$ is a scalar multiple of $\mathbf{AB}$.

$$\mathbf{AP} = t\mathbf{AB} \qquad \text{where } t \text{ is a real number.}$$

This gives $\mathbf{p} = \mathbf{a} + \mathbf{AP} = \mathbf{a} + t\mathbf{AB} = \mathbf{a} + t(\mathbf{b} - \mathbf{a})$

IMPORTANT

The **vector equation** of the line through points A and B with position vectors is:

$$\mathbf{p} = \mathbf{a} + t\mathbf{AB} \text{ (for some real number } t\text{)}$$

Alternatively, if $\mathbf{u}$ is any vector in the direction of the line AB and if $\mathbf{w}$ is any point on the line, then $\mathbf{p} = \mathbf{w} + t\mathbf{u}$. Of course, the value of t will be different for each point.

By varying the value of t, you can locate any point on the line.

- If $0 \le t \le 1$, point P lies on the **line segment** AB.
- If $t > 1$, point P lies on the line beyond B.
- If $t < 0$, point P lies on the line beyond A to the left, as shown in the previous diagram.

For $\mathbf{u}$ parallel to $\mathbf{AB}$, $\mathbf{p} = \mathbf{a} + t\mathbf{u}$ locates any point on the line AB. The choice of A is arbitrary, as long as it is on the line.

Example 16

a Find the vector equation of the line which passes through the points $A(3, 1)$ and $B(5, -2)$

b Find the vector equation of the line segment from $C(2, -3, 5)$ to $D(-1, 7, 4)$.

Solution

a Find **AB**.	$\mathbf{AB} = (5\mathbf{i} - 2\mathbf{j}) - (3\mathbf{i} + \mathbf{j})$
Simplify.	$= 2\mathbf{i} - 3\mathbf{j}$
Write the vector equation.	$\mathbf{p} = \mathbf{a} + t\mathbf{AB}$
Substitute values.	$\mathbf{p} = 3\mathbf{i} + \mathbf{j} + t(2\mathbf{i} - 3\mathbf{j})$

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b Find **CD.**	$\mathbf{CD} = (2\mathbf{i} - 3\mathbf{j} + 5\mathbf{k}) - (-\mathbf{i} + 7\mathbf{j} + 4\mathbf{k})$
Simplify.	$= 3\mathbf{i} - 10\mathbf{j} + \mathbf{k}$
Write the vector equation.	$\mathbf{p} = \mathbf{c} + t\mathbf{CD}$
Substitute values and restrict t.	$\mathbf{p} = -\mathbf{i} + 7\mathbf{j} + 4\mathbf{k} + t(3\mathbf{i} - 10\mathbf{j} + \mathbf{k})$ for $0 \le t \le 1$

Sometimes it is useful to convert the vector form of the equation of a line into the Cartesian form. You do this by eliminating the parameter t. Consider the points $A(a_1, a_2, a_3)$, $B(b_1, b_2, b_3)$ and $P(x, y, z)$ lying on the line AB.

Then $\mathbf{p} = \mathbf{a} + t\mathbf{AB} = \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} + t\begin{bmatrix} b_1 - a_1 \\ b_2 - a_2 \\ b_3 - a_3 \end{bmatrix} = \begin{bmatrix} a_1 + t(b_1 - a_1) \\ a_2 + t(b_2 - a_2) \\ a_3 + t(b_3 - a_3) \end{bmatrix}$

So

$$x = a_1 + t(b_1 - a_1)$$

$$y = a_2 + t(b_2 - a_2)$$

$$z = a_3 + t(b_3 - a_3)$$

Rearranging these equations gives:

$$t = \frac{x - a_1}{b_1 - a_1}$$

$$t = \frac{y - a_2}{b_2 - a_2}$$

$$t = \frac{z - a_3}{b_3 - a_3}$$

Equating these expressions gives:

$$\frac{x - a_1}{b_1 - a_1} = \frac{y - a_2}{b_2 - a_2} = \frac{z - a_3}{b_3 - a_3}$$

IMPORTANT

The **Cartesian form** of the equation of a line in space passing through the points $A(a_1, a_2, a_3)$ and $B(b_1, b_2, b_3)$ is:

$$\frac{x - a_1}{b_1 - a_1} = \frac{y - a_2}{b_2 - a_2} = \frac{z - a_3}{b_3 - a_3}$$

You can also use the form $\mathbf{p} = \mathbf{a} + t\mathbf{u}$, where $\mathbf{u}$ is any vector in the direction of the line. In this case, you get $x = a_1 + tu_1$, etc, so the vector equation can be written as follows.

IMPORTANT

The vector equation of a line is $\mathbf{p} = \mathbf{a} + t\mathbf{u}$, where $\mathbf{u} = u_1\mathbf{i} + u_2\mathbf{j} + u_3\mathbf{k}$ is a vector in the direction of the line and $\mathbf{a} = a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k}$ is a point on the line.

The corresponding Cartesian form is $\dfrac{x-a_1}{u_1} = \dfrac{y-a_2}{u_2} = \dfrac{z-a_3}{u_3}$.

The **parametric form** of a straight line was given in the derivation above: $x = a_1 + t(b_1 - a_1)$, $y = a_2 + t(b_2 - a_2)$, $z = a_3 + t(b_3 - a_3)$. Since $b_1 - a_1$, $b_2 - a_2$ and $b_3 - a_3$ are constants, this can be shown as $x = a_1 + c_1t$, $y = a_2 + c_2t$, $z = a_3 + c_3t$. In the two dimensional case of $x = a_1 + c_1t$, $y = a_2 + c_2t$, elimination of t gives $\dfrac{x-a_1}{c_1} = \dfrac{y-a_2}{c_2}$ or $y = \dfrac{c_2}{c_1}x + a_2 - \dfrac{a_1c_2}{c_1}$ so the slope is $\dfrac{c_2}{c_1}$ and the y-intercept is $a_2 - \dfrac{a_1c_2}{c_1}$. If the parametric form of a line is not linear (in the parameter) in at least one dimension, then the line will not be straight. For example, the simple 2-dimensional case $x = t + 1$ and $y = 4t^2 + 3$ is the parametric form of $y = 2(x - 1)^2 + 3$.

Example 17

Find the Cartesian form of the equation of the straight line which passes through the points (8, 4,–2) and (5, 2,–1).

Solution

Assign the points.	Let $A(a_1, a_2, a_3) = (8, 4, -2)$ and $B(b_1, b_2, b_3) = (5, 2, -1)$.
Write the Cartesian form of the equation.	$\dfrac{x-a_1}{b_1-a_1} = \dfrac{y-a_2}{b_2-a_2} = \dfrac{z-a_3}{b_3-a_3}$
Substitute for $a_1, a_2, \ldots$	$\dfrac{x-8}{5-8} = \dfrac{y-4}{2-4} = \dfrac{z-(-2)}{-1-(-2)}$
Simplify.	$\dfrac{x-8}{-3} = \dfrac{y-4}{-2} = \dfrac{z+2}{1}$

The vector equation for the position of an object moving with constant velocity in a straight line is given by $\mathbf{p} = \mathbf{a} + \mathbf{v}t$, where $\mathbf{a}$ is the position when $t = 0$ and $\mathbf{v}$ is the velocity. This is the vector equation of a straight line, so you can use this to determine if the paths of two objects would cross, or even if they would collide.

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Example 18

a Find the vector equation of the line that passes through the point $A(1, -1, 0)$ that is parallel to the line passing through $M(-3, 2, 1)$ and $N(2, 1, 0)$.

b Show that $B(-9, 1, 2)$ lies on the line.

c Two objects are moving with constant velocity in straight lines. The first starts at $(0, 2, -1)$ and has velocity $(1, -1, 0)$ and the second has velocity $(0, 1, -1)$, starting from $(2, -3, 2)$. Do their paths cross and do they collide?

Solution

a Find the vector **MN**.	$\mathbf{MN} = (2\mathbf{i} + \mathbf{j}) - (-3\mathbf{i} + 2\mathbf{j} + \mathbf{k})$
Simplify.	$= 5\mathbf{i} - \mathbf{j} - \mathbf{k}$
Write the vector equation for the line.	$\mathbf{p} = \mathbf{a} + t\mathbf{u}$
Substitute.	$= \mathbf{i} - \mathbf{j} + t(5\mathbf{i} - \mathbf{j} - \mathbf{k})$
b Write the position vector **b**.	$\mathbf{b} = -9\mathbf{i} + \mathbf{j} + 2\mathbf{k}$
Write an equation for the value of t.	Let $-9\mathbf{i} + \mathbf{j} + 2\mathbf{k} = (\mathbf{i} - \mathbf{j}) + t(5\mathbf{i} - \mathbf{j} - \mathbf{k})$
Equate **i** parts.	$-9 = 1 + 5t$ so $t = -2$
Check $t = 2$ in the other equations.	
Equate **j** parts.	$1 = -1 + (-2)(-1) = 1$ ✓
Equate **k** parts.	$2 = 0 + (-2)(-1) = 2$ ✓
State the result.	$B(-9, 1, 2)$ lies on the line that passes through $A(1, -1, 0)$, which is parallel to the line passing through $M(-3, 2, 1)$ and $N(2, 1, 0)$.
c Write the position of the first object.	Position of first, $\mathbf{p}_1 = 2\mathbf{j} - \mathbf{k} + t(\mathbf{i} - \mathbf{j})$
Write the position of the second object.	Position of second, $\mathbf{p}_2 = 2\mathbf{i} - 3\mathbf{j} + 2\mathbf{k} + t(\mathbf{j} - \mathbf{k})$
Write the condition for the paths to cross.	The paths cross if $\mathbf{p}_1 = \mathbf{p}_2$ at times t and s, say.
Equate the **i** parts of $\mathbf{p}_1$ and $\mathbf{p}_2$.	$t\mathbf{i} = 2\mathbf{i} + 0\mathbf{i}$, so $t = 2$
Substitute in the first equation.	$\mathbf{p}_1 = 2\mathbf{j} - \mathbf{k} + 2(\mathbf{i} - \mathbf{j}) = 2\mathbf{i} - \mathbf{k}$
Equate the **j** parts with time s in $\mathbf{p}_2$.	$0\mathbf{j} = -3\mathbf{j} + s\mathbf{j}$ so $s = 3$
Substitute in the second equation.	$\mathbf{p}_2 = 2\mathbf{i} - 3\mathbf{j} + 2\mathbf{k} + 3(\mathbf{j} - \mathbf{k}) = 2\mathbf{i} - \mathbf{k}$
State the result.	$2\mathbf{i} - \mathbf{k}$ is on both lines so the paths cross.
Write the times where the paths cross.	The first passes through $(2, 0, -1)$ at $t = 2$ and the second at $t = 3$.
Write the overall answer.	Their paths cross but they do not collide.

The method shown in part **c** above can be used to determine whether lines meet. It they are not parallel but do not meet, then they are **skew** lines.

EXERCISE 1.06 Lines in 3D

Concepts and techniques

WS Equations of lines in space

1 Example 16 a Write down the vector **AB** that joins the points $A(3, 2, 7)$ and $B(-1, 2, 3)$.
b Find the vector equation of the line through A and B.
c How is the equation of the line segment AB different to the equation of the line AB?

2 Write down the vector equation of the line segment between the points with position vectors $\mathbf{p} = 3\mathbf{i} + 7\mathbf{j} - 2\mathbf{k}$ and $\mathbf{q} = -3\mathbf{i} + 2\mathbf{j} + 2\mathbf{k}$. Also, find the Cartesian equation of this line.

3 Find the vector equation of the line passing through $(9, 1, 2)$, which is parallel to the vector $\mathbf{u} = \mathbf{i} + \mathbf{j} + \mathbf{k}$.

4 Example 17 Find the Cartesian equation of the line passing through the origin and $(1, -5, 3)$.

5 Find the Cartesian equation of the line passing through $(1, 0, 1)$ and $(0, 1, 1)$.

6 Find the Cartesian form of the equation passing through the point $(1, 2, 1)$, which is parallel to $\mathbf{u} = 4\mathbf{i} + 5\mathbf{j} - \mathbf{k}$.

7 Find the vector equation of the line whose Cartesian equation is $\frac{x}{3} = \frac{y-1}{1} = \frac{z+1}{2}$.

8 A line passes through the point $(2, 3, 5)$ in the direction of $\mathbf{i} + 3\mathbf{j} - \mathbf{k}$. Find:
a the vector equation of the line
b the Cartesian equation of the line.

9 If $A = (1, 2, 3)$, $B = (-1, -2, -3)$ and $C = (4, -4, 4)$, find the Cartesian equation of:
a AB
b AC

10 Find the vector and Cartesian equations of the line passing through A, in the direction of $\mathbf{u}$ for each of the following.
a $A(1, 0, -1)$ and $\mathbf{u} = 2\mathbf{i} + 2\mathbf{j} - \mathbf{k}$
b $A(0, 1, -1)$ and $\mathbf{u} = \mathbf{i} + 2\mathbf{j}$
c $A(-6, 5, 2)$ and $\mathbf{u} = 2\mathbf{i} - 5\mathbf{k}$
d $A(2, 3, -3)$ and $\mathbf{u} = -\mathbf{i} + 3\mathbf{k}$

11 Find the vector and Cartesian equations of the line passing through A and B in each of the following cases.
a $A(-4, 3, 5)$ and $B(-2, 4, 1)$
b $A(0, 5, 0)$ and $B(5, 0, -5)$
c $A(9, 3, -2)$ and $B(4, 5, -1)$
d $A(2, -1, 3)$ and $B(1, 4, -3)$

12 Example 18 The Cartesian equation of a line is given by: $\frac{x-1}{1} = \frac{y}{2} = \frac{z-1}{3}$.

Do the points $P(3, 2, 6)$ and $Q(-1, -4, -5)$ lie on the line?

13 Given the points $P(1, -1, 0)$, $M(-3, 2, 1)$ and $N(2, 1, 0)$:
a find the vector equation of the line that passes through P and is parallel to MN
b show that $Q(-14, 2, 3)$ lies on the line determined in part **a**.

Reasoning and communication

14 Find the Cartesian form of the lines given by the following parametric equations and describe the curve.

a $x = t - 4$, $y = t^3 - 2$

b $x = \cos(t)$, $y = \sin(t)$ [Hint: use the Pythagorean identity]

c $x = 4\cos(t)$, $y = 3\sin(t)$

15 Two lines have the vector equations:

$\mathbf{p} = \mathbf{i} + \mathbf{j} - 2\mathbf{k} + t(\mathbf{j} - \mathbf{k})$

$\mathbf{q} = 2\mathbf{j} - 2\mathbf{k} + s(\mathbf{i} - \mathbf{j})$

Find the point of intersection of the lines.

16 Given that the position vectors of A, B, C and D are $\mathbf{a} = \mathbf{i} - 2\mathbf{j} + \mathbf{k}$, $\mathbf{b} = -\mathbf{i} - 3\mathbf{k}$, $\mathbf{c} = 3\mathbf{i} + \mathbf{j} - 2\mathbf{k}$ and $\mathbf{d} = 8\mathbf{i} + \mathbf{j} + 4\mathbf{k}$ respectively, find the vector equation of the line:

a through A and B

b through D and parallel to BC

c through C and parallel to AB.

17 Consider the following vector equations.

Line 1: $\mathbf{p} = 7\mathbf{i} - 3\mathbf{j} + 3\mathbf{k} + t(3\mathbf{i} - 2\mathbf{j} + \mathbf{k})$

Line 2: $\mathbf{q} = 7\mathbf{i} - 2\mathbf{j} + 4\mathbf{k} + s(-2\mathbf{i} + \mathbf{j} - \mathbf{k})$

Line 3: $\mathbf{r} = \mathbf{i} + u(\mathbf{j} - \mathbf{k})$

a Show that Line 1 and Line 2 intersect and find their point of intersection.

b Show that Line 1 and Line 3 are skew lines.

c Determine if Line 2 and Line 3 intersect, are parallel or are skew lines.

18 A line passes through (1, 2, 3) and is parallel to $\mathbf{d} = 2\mathbf{i} - \mathbf{j} - 4\mathbf{k}$.

a Find the vector and Cartesian equations of the line.

b Which of the points $P(2, 0, 1)$, $Q(3, 1, -1)$ and $R(\frac{1}{2}, 2\frac{1}{4}, 4)$ lie on the line?

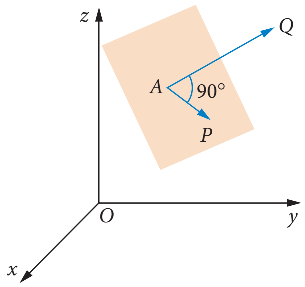
19 At what point does the line passing through (3, 1, –2) and (–2, 7, –4) intersect the xy-plane?

20 Two particles begin moving from positions (5, 7, –9) and (–4, –2, 9) with velocities (1, 2, 3) and (2, 3, 1) respectively. Determine if their paths cross, and if so, whether they collide.

21 Two aeroplanes close to an airport are moving at speeds of 306 km/h and 396 km/h respectively. The first plane is descending at a rate of 31 m/s on a SE path and the second is rising at 13 m/s on a NE path. At 9:30 p.m. the air traffic controller plots their positions as 3.8 km N and 1.1 km E of the airport and 2.2 km S and 0.55 km E of the control tower with altitudes of 6972 m and 5031 m respectively. Will they get dangerously close to each other (within 500 m)?

22 One aeroplane is travelling near an airport in a path given by $x = 10\cos(6t)$, $y = 10\sin(6t)$, $z = 6$ where x, y and z are in km and the airport is at (0, 0, 0). Another plane is travelling on the path $x = 12\cos(5t)$, $y = 12\sin(5t)$, $z = 8$ while a third plane is following the path $x = 40(t - 0.1)$, $y = 60(t - 0.1)$, $z = 0.1 - t$. z is the vertical axis, x is north, y is east and t is time in hours. Describe the paths and what the aircraft are doing.

1.07 EQUATIONS OF PLANES



There are a number of ways of specifying a plane. Consider the points $P(x, y, z)$, $A(a_1, a_2, a_3)$ and Q, where AQ is perpendicular to the plane.

In component form, $\mathbf{AP} = (x - a_1, y - a_2, z - a_3)$

Let $\mathbf{AQ} = \mathbf{n} = (n_1, n_2, n_3)$

Because $\mathbf{n}$ and $\mathbf{AQ}$ are orthogonal, their scalar product is zero:

$$\mathbf{n} \cdot \mathbf{AP} = 0$$

So $(n_1, n_2, n_3) \cdot (x - a_1, y - a_2, z - a_3) = 0$

$n_1(x - a_1) + (n_2(y - a_2) + n_3(z - a_3) = 0$

IMPORTANT

The **Cartesian equation of a plane** through $A(a_1, a_2, a_3)$, with normal vector $\mathbf{n} = (n_1, n_2, n_3)$ is:

$$n_1(x - a_1) + n_2(y - a_2) + n_3(z - a_3) = 0$$

Let the position vectors of P and A be $\mathbf{p}$ and $\mathbf{a}$ respectively, then, $\mathbf{AP} = \mathbf{p} - \mathbf{a}$.

$\mathbf{n}$ and $\mathbf{AP}$ are orthogonal, so:

$$\mathbf{n} \cdot \mathbf{AP} = 0$$

or $\quad \mathbf{n} \cdot (\mathbf{p} - \mathbf{a}) = 0$

IMPORTANT

The **vector equation of the plane** through $A(a_1, a_2, a_3)$, with normal vector $\mathbf{n}$ is:

$\mathbf{n} \cdot (\mathbf{p} - \mathbf{a}) = 0$, where P is a point in the plane.

This equation may also be written as

$$\mathbf{p} \cdot \mathbf{n} = \mathbf{a} \cdot \mathbf{n} = k$$

where k is a constant.

A plane is often represented by a capital Greek letter such as Γ (gamma). A **region of a plane** can be specified by adding additional conditions to the equation of the plane.

Example 19

Consider the plane, Γ, passing through $A(2, -2, 4)$ with normal $\mathbf{n} = (1, 2, 3)$.

a Find the Cartesian equation of the plane.

b Does the point $B(-2, 3, 2)$ lie on the plane?

c Find an expression for the points inside the circle on the plane with centre A and radius 10.

Solution

a Write the equation of the plane. $\mathbf{n} \cdot (\mathbf{p} - \mathbf{a}) = 0$

so $n_1(x - a_1) + n_2(y - a_2) + n_3(z - a_3) = 0$

Substitute in the values. $1(x - 2) + 2(y + 2) + 3(z - 4) = 0$

Simplify. $x + 2y + 3z - 10 = 0$

b Write the LHS of the equation. $\text{LHS} = x + 2y + 3z - 10$

Substitute values for B. $= -2 + 2 \times 3 + 3 \times 2 - 10$

Evaluate. $= 0 = \text{RHS}$

Write the answer. $A(-2, 3, 2)$ lies on the plane

$x + 2y + 3z - 10 = 0$

c Write the distance of the points inside the circle from A. $\sqrt{(x-2)^2 + [y-(-2)]^2 + (z-4)^2} < 10$

Square both sides. $(x - 2)^2 + (y + 2)^2 + (z - 4)^2 < 100$

Rewrite the equation of Γ. But $x = 10 - 2y - 3z$ for points on Γ.

Substitute in the inequality. $(10 - 2y - 3z - 2)^2 + (y + 2)^2 + (z - 4)^2 < 100$

Simplify the first bracket. $(8 - 2y - 3z)^2 + (y + 2)^2 + (z - 4)^2 < 100$

Multiply out and collect terms. $5y^2 + 10z^2 + 12yz - 28y - 56z + 84 < 100$

Write the answer. The points satisfy the inequality

$5y^2 + 10z^2 + 12yz - 28y - 56z < 16$

In Example **19** part **c**, the answer could be given in terms of x and y or in terms of x and z instead of in terms of y and z. The alternate solutions are $10x^2 + 13y^2 + 4xy - 32x + 44y < 824$ or $5x^2 + 12z^2 - 4xz - 44x - 72z < 124$. Non-circular regions of an arbitrary plane could also be found, but they are beyond the scope of this course.

The equation of a plane can be written as $\mathbf{p} \cdot \mathbf{n} = \mathbf{a} \cdot \mathbf{n} = k$

If $\mathbf{n} = (l, m, n)$ in component form, then the equation of the plane becomes:

$$\mathbf{p} \cdot \mathbf{n} = (x, y, z) \cdot (l, m, n) = k$$

so $lx + my + nz = k$, which is an alternative Cartesian equation for the plane.

IMPORTANT

The graph of the equation $lx + my + nz = k$ is a plane with normal $\mathbf{n} = (l, m, n)$.

Example 20

Find the equation of the plane, Γ, through $A(3, -5, 4)$, which is parallel to the plane $3x + y - 6z + 8 = 0$.

Solution

Write the normal to the plane.	$lx + my + nz = k$ has the normal $\mathbf{n} = (l, m, n)$.
Write the normal for this plane.	$3x + y - 6z + 8 = 0$ has the normal $\mathbf{n} = (3, 1, -6)$.
Write the equation of Γ.	The equation of Γ is $3x + y - 6z = k$
Substitute $A(3, -5, 2)$ in the LHS.	$\text{LHS} = 3 \times 3 + (-5) - 6 \times 4$
Simplify.	$= -20$
Use the answer.	But A is on the plane, so $-20 = \text{RHS} = k$
State the answer.	The equation of Γ is $3x + y - 6z + 20 = 0$

You can find the equation of a plane from any three non-collinear points, as they determine a plane. For 3 non-collinear points A, B and C, the cross product of two sides of the triangle will give a normal, which can then be used with any point to determine the plane.

Example 21

Find the equation of the plane that contains the points $P(0, -2, 1)$, $Q(4, 1, 3)$ and $R(-1, 0\ 2)$.

Solution

Use P and R to find **PR**.	$\mathbf{PR} = (-1, 0, 2) - (0, -2, 1)$ $= (-1, 2, 1)$
Use P and Q to find **PQ**.	$\mathbf{PQ} = (4, 1, 3) - (0, -2, 1)$ $= (4, 3, 2)$
Find a normal vector.	$\mathbf{n} = \mathbf{PR} \times \mathbf{PQ}$
Substitute values.	$= (-1, 2, 1) \times (4, 3, 2)$ $= (2 \times 2, 1 \times 4, -1 \times 3) - (1 \times 3, -1 \times 2, 2 \times 4)$ $= (4, 4, -3) - (3, -2, 8)$
Calculate the answer.	$= (1, 6, -11)$ $= \mathbf{i} + 6\mathbf{j} - 11\mathbf{k}$
Write the Cartesian equation of the plane.	$n_1(x - a_1) + n_2(y - a_2) + n_3(z - a_3) = 0$
Use $(0, -2, 1)$ and $\mathbf{n} = (1, 6, -11)$.	$1(x - 0) + 6(y + 2) - 11(z - 1) = 0$
Simplify.	$x + 6y - 11z + 23 = 0$
State the result.	The plane has the equation $x + 6y - 11z + 23 = 0$

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EXERCISE 1.07 Equations of planes

Concepts and techniques

WS Equations of planes

1 Example 19 Which plane does not contain the point $P(10, -3, 5)$?

A $3x + 6y - 2z - 2 = 0$
B $2x - 2y - 3z - 11 = 0$
C $x + y - z - 12 = 0$
D $4x + 5y + z - 30 = 0$
E $2x - 2y - 3z - 11 = 0$

2 The equation $5x + 3y - 7 = 0$ can represent

A a Cartesian equation of a line in two dimensions
B a Cartesian equation of a plane in three-dimensional space
C a vector equation of a plane in three-dimensional space
D neither **A** nor **B**
E both **A** and **B**

3 Determine if the points $P(1, 3, -1)$, $Q(3, 5, 1)$, and $R(1, 6, 5)$ are on the plane defined by the equation $x + 2y - z - 8 = 0$.

4 Which of the following points lie on the plane $4x + 3y - 5z - 10 = 0$?

a $A(1, 2, 0)$
b $B(-7, 6, 4)$
c $C(-2, 1, -3)$
d $D(1.2, -2.4, 6.2)$

5 a Find the equation of the plane that contains the point $A(3, -1, 0)$, for which the vector $\mathbf{r} = 2\mathbf{i} + 10\mathbf{j} - 7\mathbf{k}$ is perpendicular to the plane.

b Find an expression for the points inside the circle on the plane with centre $(2, 2, 4)$ and radius 6.

c Find an expression for the points on the plane on or outside the circle on the plane with centre A and radius 9.

6 a Find the equation of the plane that contains the point $(0, 1, 2)$ and is perpendicular to $\mathbf{n} = 2\mathbf{i} + \mathbf{j} - 3\mathbf{k}$.

b Find an expression for the points on or inside the circle on the plane with centre $(0, 1, 2)$ and radius 8.

c Find an expression for the points on the plane outside the circle on the plane with centre $(1, 2, 3)$ and radius 5.

7 Find the equation of the plane through the point $(5, -2, 4)$ with normal vector $\mathbf{n} = \mathbf{i} + 2\mathbf{j} + 3\mathbf{k}$.

8 Find the equation of the plane passing through $(0, 0, 0)$, with normal vector $2\mathbf{i} + 4\mathbf{j} - 5\mathbf{k}$.

9 A plane passes through the point $P(3, 2, 5)$ and is perpendicular to $\mathbf{n} = \mathbf{i} + \mathbf{k}$.

a Find the vector equation of the plane.
b Find the Cartesian equation of the plane.

10 Find the equation of the plane that passes through $D(5, 2, -3)$ and is orthogonal to the planes $2x - y + 4z = 7$ and $3y - z = 8$.

11 Example 20 Which vector is not normal to the plane $x + 2y - 3z - 4 = 0$?

A $(-4, -8, 12)$
B $(2, -3, 4)$
C $(2, 4, -6)$
D $(3, 6, -9)$
E $(-1, -2, 3)$

12 Find the equation of the plane that passes through $A(5, -2, 3)$ and is parallel to the plane $3x + y - 6z + 8 = 0$.

13 Find the equation of the plane that passes through $M(-11, 4, -2)$ with normal vector $\mathbf{n} = 6\mathbf{i} - 5\mathbf{j} - \mathbf{k}$.

14 Find the equation of the plane through $D(5, -3, 2)$ that is:

a parallel to the xy-plane b parallel to the yz-plane c parallel to the xz-plane

15 Find the equation of the plane that passes through $Q(2, 5, -6)$ and is parallel to the plane $3x - y + 2z = 10$.

16 Find the equation of the plane passing through the point $(1, 1, 1)$ and is parallel to the plane $2x + y + z = 0$.

17 Example 21 Find the equation of the plane that contains the points in each of the following cases.

a $(1, -2, 0)$, $(0, -1, 2)$ and $(3, 1, 4)$

b $(1, 0, 1)$, $(2, 4, 6)$ and $(1, 2, -1)$

c $(1, -2, -3)$, $(4, -4, 4)$ and $(3, 2, -3)$

d $(0, 1, 2)$, $(2, 1, 3)$ and $(3, 1, 0)$

e $(0, 0, 0)$, $(0, 2, 5)$ and $(1, 4, 0)$

f $(1, 0, 2)$, $(-1, 1, 2)$ and $(5, 0, 3)$

18 Find the equation of the plane that contains the points $A(2, -1 , 0)$, $B(-1 , 3, 4)$ and $C(3, 0, 2)$.

19 Given the points $A(1, 0, 0)$, $B(1, 2, 2)$ and $C(3, 0, 1)$, find the equation of the plane passing through A, B, and C.

20 Find the equation of the plane passing through the points $(0, -1, 1)$, $(1, 0, 1)$ and $(1, 2, 2)$.

Reasoning and communication

21 A plane passes through $(7, 3, -5)$ and the vector normal to the plane, $\mathbf{n} = \mathbf{i} + \mathbf{j} + \mathbf{k}$. Find the shortest distance from the plane to the origin.

22 What is the distance of the plane $\mathbf{p} \cdot (3\mathbf{i} + 2\mathbf{j} + \mathbf{k}) = 5$ to the origin?

23 Determine if the planes $-6x + 9y + 3z + 2 = 0$ and $2x - 3y - z - 5 = 0$ are parallel, perpendicular or neither.

24 Find the distance from the point $Q(-5, 3, 2)$ to the plane $4x - 5y + 8z - 7 = 0$.

25 Find the equation of the plane through the point $P(1, 2, -3)$ and the line of intersection of the planes $x + 3y - 5z - 10 = 0$ and $6x - y + z = 0$.

26 Determine whether the following pair of planes $\Gamma 1$ and $\Gamma 2$ are parallel or intersect.

$\Gamma 1 : x + 2y - 4z - 2 = 0$

$\Gamma 2 : -2x - 4y + 8z - 1 = 0$

27 The following equations describe a set of planes. Describe how the planes intersect.

$\Gamma 1$: $x - 5y + 2z - 10 = 0$

$\Gamma 2$: $x + 7y - 2z + 6 = 0$

$\Gamma 3$: $8x + 5y + z - 20 = 0$

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1.08 SYSTEMS OF LINEAR EQUATIONS

A **linear equation** is one that can be written with no powers, roots, reciprocals, and so on—for example, $5x - y + 4z = 8$. A **system** of equations shows the relationships between a group of variables. You already know how to solve simultaneous equations with two variables.

The main steps used to solve simultaneous equations by elimination (for two equations) are as follows.

1 Choose which variable to eliminate.
2 Make the coefficients of that variable the same size, but opposite in sign, by multiplying by suitable numbers.
3 Add the equations to eliminate the variable.

Once the variable is eliminated, you can divide to find the value of the remaining variable and substitute back to find the first variable. This method can be extended to multiple equations, proceeding to eliminate one variable at a time. The elimination of one variable at a time, reducing both the number of variables and the number of equations at each step, is usually called **Gaussian elimination**.

IMPORTANT

Gaussian elimination steps for multiple equations

1 Choose which variable to eliminate.

2 Put the equation with the simplest coefficient of this variable first in the list.

3 For each of the other equations, multiply the chosen equation by a suitable number and then add to eliminate the chosen variable.

4 Repeat steps 1–3 with the new equations until an equation with a single variable is obtained, if possible. Find its value.

5 Progressively substitute to find the values of all the variables.

Example 22

Solve the following set of equations by elimination.

$3a - 2b - 4c = 4$

$2a + b + 3c = 4$

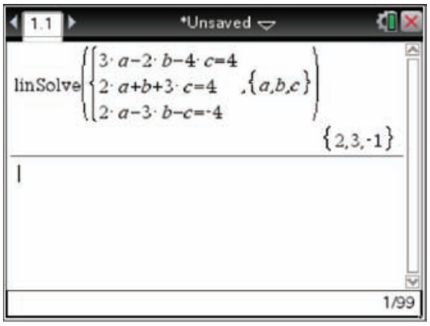
$2a - 3b - c = -4$

Solution

Put the simplest equation first.	$2a + b + 3c = 4$ $3a - 2b - 4c = 4$ $2a - 3b - c = -4$	[2] [1] [3]
In [1] and [2], make the coefficients of *b* the same but with opposite signs. Add the equations to eliminate *b*.	$4a + 2b + 6c = 8$ $3a - 2b - 4c = 4$ $7a \quad + 2c = 12$	$[2] \times 2$ [1] [4]
In [1] and [3], make the coefficients of *b* the same but with opposite signs. Add the equations to eliminate *b*.	$6a + 3b + 9c = 12$ $2a - 3b - c = -4$ $8a \quad + 8c = 8$	$[2] \times 3$ [3]
Simplify.	$a + c = 1$	[5]
In [5] and [4], make the coefficients of *c* the same but with opposite signs. Add the equations to eliminate *c*.	$-2a - 2c = -2$ $7a + 2c = 12$ $5a \quad = 10$	$[5] \times -2$ [4]
Simplify. Substitute in [5]. Solve for *c*.	$a = 2$ $2 + c = 1$ $c = -1$	[6]
Now substitute in [2]. Solve for *b*.	$2 \times 2 + b + 3 \times (-1) = 4$ $4 + b - 3 = 4$ $b = 3$	
Check that the solution works in all the original equations.	For $a = 2$, $b = 3$ and $c = -1$: $3a - 2b - 4c = 3 \times 2 - 2 \times 3 - 4 \times (-1) = 4$ [1]✓ $2a + b + 3c = 2 \times 2 + 3 + 3 \times (-1) = 4$ [2]✓ $2a - 3b - c = 2 \times 2 - 3 \times 3 - (-1) = -4$ [3]✓	
Write the answer.	$a = 2$, $b = 3$ and $c = -1$	

TI-Nspire CAS

Use [menu], 3: Algebra, 7: Solve System of Equations and 2: Solve System of Linear Equations.
Make the number of variables 3 and enter *a*, *b* and *c* as the variables. Enter the equations and press [enter].



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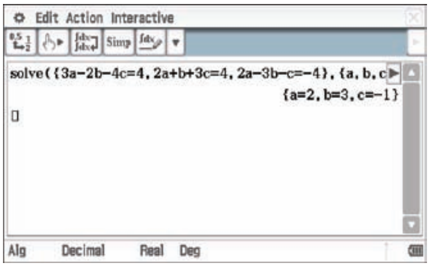
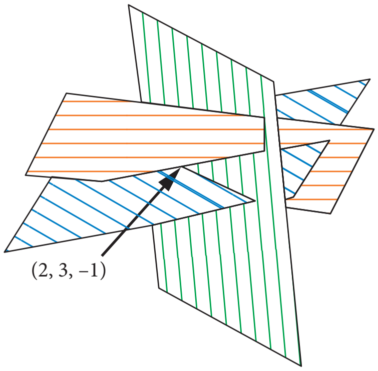
ClassPad

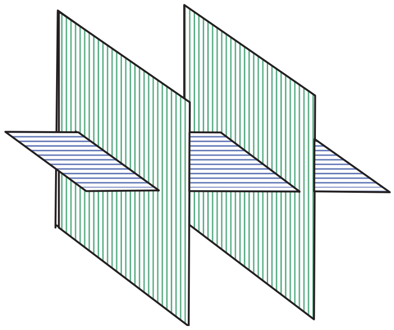
Tap **Action** then **Equation/Inequality**, then **solve**. To enter the list of equations, first tap the braces {} key under Keyboard Math3. Between the braces, enter the three equations with a comma after each one. Press , and in a second pair of braces, list the variables a,b,c as shown and close the parentheses on the solve command. Press EXE to solve the system of equations for *a*, *b* and *c*.



What does Example **22** mean? $3a - 2b - 4c = 4$, $2a + b + 3c = 4$ and $2a - 3b - c = -4$ are all the equations of planes. The point $(2, 3, -1)$ is on all three planes, so they intersect at one point, as illustrated on the right. These have a *unique solution*.



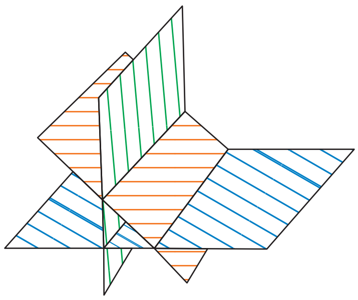
What other possibilities are there? Two planes could be parallel, as shown on the right. There would be no point of intersection so there would be no solution.



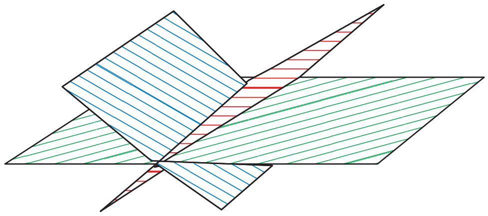
Three parallel planes have no solution either. They have the same normal vector but no common points, such as $3x + 4y - 2z = 5$, $3x + 4y - 2z = 10$ and $6x + 8y - 4z = -5$.

Likewise, three planes that intersect in parallel lines as shown on the right have no point of intersection.

Equations like this give an absurd result when you try to solve them. If you subtract the first equation from the second equation, you get $0 = 5$.



You could have 3 planes intersecting in a line as shown here.



These have an infinite number of solutions corresponding to points on the line. When you try to solve equations for planes like this, such as $x + 2y - 2z + 1 = 0$, $4x - y - 2z + 4 = 0$ and $3x + 3y - 4z + 3 = 0$, you can only eliminate one variable. One of the equations can be written as a combination of the other two because the normal vectors are in the same plane.

IMPORTANT

There are three cases that occur for simultaneous equations.

Inconsistent equations correspond to cases involving at least a pair of parallel planes or planes that intersect in parallel lines.

A **unique solution** corresponds to planes that intersect at a single point.

Dependent equations have an equation that can be written as a combination of the others. This corresponds to planes that intersect in a straight line.

Similar considerations apply to equations with four or more unknowns corresponding to multidimensional vector spaces, but these are beyond the scope of this course. In any case, simultaneous equations with many variables are normally solved using approximate matrix methods to obtain 'good enough' solutions.

Example 23

Solve the following by elimination.

$$x + 2y - 2z + 1 = 0$$
$$4x - y - 2z + 4 = 0$$
$$3x + 3y - 4z + 3 = 0$$

Solution

Put the simplest equation first.	$4x - y - 2z + 4 = 0$ $x + 2y - 2z + 1 = 0$ $3x + 3y - 4z + 3 = 0$	[2] [1] [3]
In [1] and [2], make the coefficients of y the same but with opposite signs. Add the equations to eliminate y.	$8x - 2y - 4z + 8 = 0$ $x + 2y - 2z + 1 = 0$ $9x \quad - 6z + 9 = 0$	[2] × 2 [1] [4]
Simplify.	$3x - 2z + 3 = 0$	
In [1] and [3], make the coefficients of y the same but with opposite signs. Add the equations to eliminate y.	$12x - 3y - 6z + 12 = 0$ $3x + 3y - 4z + 3 = 0$ $15x \quad - 10z + 15 = 0$	[2] × 3 [3] [5]
Simplify.	$3x - 2z + 3 = 0$	[5]

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State the situation. [4] and [5] are the same.

Choose a value for z. Let $z = c$, where $c \in \boldsymbol{R}$.

Substitute in equation [4] or [5].

$$3x - 2c + 3 = 0$$

Solve for x.

$$x = \frac{2c-3}{3}$$

Substitute in equation [2].

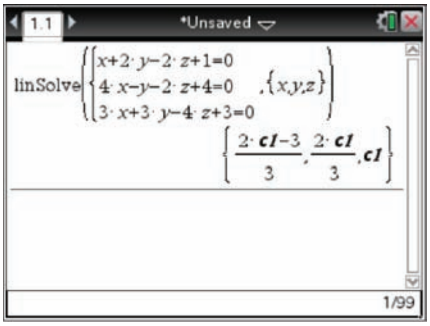
$$4 \times \frac{2c-3}{3} - y - 2c + 4 = 0$$

Solve for y.

$$y = \frac{2c}{3}$$

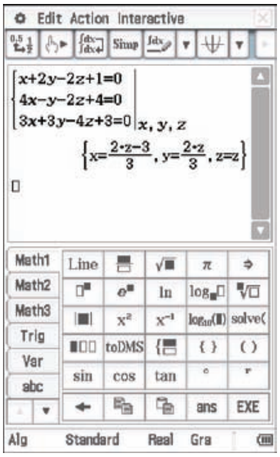
TI-Nspire CAS

Enter the equations as before.
The equations do not give a unique solution.
c1 can be any real number.



ClassPad

Ensure that your calculator is set to **Standard** mode. Use the solve command and enter the equations and variables as in Example **22**.
Alternatively, press Keyboard and tap Math1, then the simultaneous equations template. Tap it again to expand the system to three equations.
Use the up arrow to enter the first line. Enter one equation on each line and use the down arrow to move down to the next line. On the third line, use the right arrow to move across to the empty box and enter the variables, with each pair separated by a comma. Press EXE when this is done to solve the system of equations for the variables specified.
The equations do not give a unique solution.
z can be any real number.



Write the answer. The solutions can be given as

$$x = \frac{2c-3}{3},\ y = \frac{2c}{3},\ z = c, \text{ where } c \in \boldsymbol{R}.$$

In Example **23**, there are infinitely many solutions. There are different ways to express them, depending on whether x, y or z is chosen as the independent variable. For example, if y is chosen as the independent variable, the solutions are $x = c - 1$, $y = c$, $z = \frac{3c}{2}$, where $c \in \boldsymbol{R}$. Your teacher may want you to work out the form of the solutions if x is chosen as the independent variable.

Example 24

Solve the following equations.

$$\begin{aligned} p + 3q + 2m &= 4 \\ 7p + 7q - 6m &= 14 \\ 4p + 5q - 2m &= 11 \end{aligned}$$

Solution

Number the equations, with the simplest one first.

$$\begin{aligned} p + 3q + 2m &= 4 && [1] \\ 7p + 7q - 6m &= 14 && [2] \\ 4p + 5q - 2m &= 11 && [3] \end{aligned}$$

In [2] and [1], make the same coefficient of p with opposite signs.
Add the equations to eliminate p.

$$\begin{aligned} -7p - 21q - 14m &= -28 && [1] \times (-7) \\ 7p + 7q - 6m &= 14 && [2] \\ \hline -14q - 20m &= -14 \end{aligned}$$

Simplify.

$$7p + 10m = 7 \quad [4]$$

Now do equations [1] and [3].
Add the equations to eliminate p.

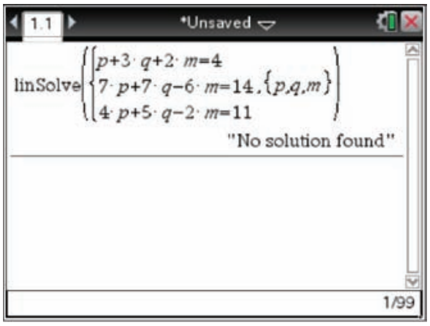
$$\begin{aligned} -4p - 12q - 8m &= -16 && [1] \times (-4) \\ 4p + 5q - 2m &= 11 && [3] \\ \hline -7q - 10m &= -5 \end{aligned}$$

Simplify.

$$7p + 10m = 5 \quad [5]$$

TI-Nspire CAS

Enter the equations as before.



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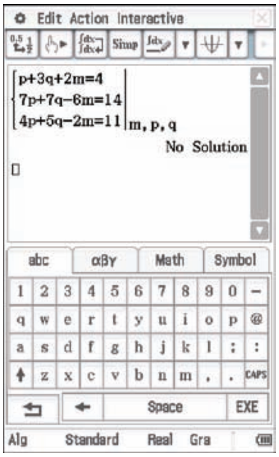
ClassPad

Use the solve command or ⊟ and enter the equations and variables as before.



Write the answer. There is no solution: the equations are inconsistent.

You should use your CAS calculator for more than 3 equations or more than 3 variables. If you have 3 variables and only two equations, then unless they are inconsistent there will be an infinite number of solutions.

EXERCISE 1.08 Systems of linear equations

Concepts and techniques

Systems of equations

1 Examples 22–24 Solve the following equations.

a
$$\begin{aligned} 7a + 2b + 4c &= 37 \\ 3a + b + 2c &= 16 \\ 17a + 5b + 11c &= 92 \end{aligned}$$

b
$$\begin{aligned} 3x + 7y + 6z &= 7 \\ 5x + 11y + 12z &= 18 \\ x + 2y + 3z &= 4 \end{aligned}$$

2 CAS Solve the following equations.

a
$$\begin{aligned} 6x + y + 2z &= 5 \\ 8x + 3y + 4z &= 5 \\ 2x - y - z &= 6 \\ 10x - 2y - z &= 19 \end{aligned}$$

b
$$\begin{aligned} 2u + 5v - 3w &= 8 \\ u - v + 4w &= 3 \\ 2u - 2v + 9w &= 9 \\ u + 2v - w &= 3 \end{aligned}$$

c
$$\begin{aligned} 4e - 8f - 10g + 5h &= 49 \\ 2e - 2f - 3g + 2h &= 17 \\ 2e - 3f - 4g + 2h &= 20 \\ 6f - 3e + 8g - 4h &= -38 \end{aligned}$$

d
$$\begin{aligned} 5b + 3c - 4d + 2e &= -17 \\ 4b + c - 3d + 2e &= -10 \\ 8d - 8b - 2c - 4e &= 24 \\ 6b + c - 6d + 3e &= -17 \end{aligned}$$

e
$$\begin{aligned} 3f - 5g + 5h - 5k &= 51 \\ 4g + 4k - 2f - 3h + 37 &= 0 \\ 6f - 11g + 10h - 12k &= 113 \\ 2f - 4g + 3h - 3k &= 32 \end{aligned}$$

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Reasoning and communication

3 Tickets for a charity performance have different prices for children, adults and seniors. When three children were taken to the performance by their grandmother, the cost was \$22. For two children and their parents, the cost was \$32, and for four children, their father and his elderly parents, the cost was \$45.
Write equations and find the price for each type of ticket.

4 CAS To make pre-mixed concrete, dry gravel, sand and cement are mixed with water in a concrete truck. Each ingredient has a different density and different mixtures are used for different purposes. Slump, for example, is a mix of gravel, sand, cement and water in the ratio 5 : 2 : 1 : 2 (by volume). This and other mixes (by volume) for standard (5 m^3) loads, and their masses, are shown in the table below. Use the information to find the density of each material.

Type	Gravel	Sand	Cement	Water	Mass (tonne)
Slump	5	2	1	2	21.4
General mix	7.2	4	2	2.8	21.125
Foundation	10	3	3	4	20.35
Fine finish	6	5	2	3	20.5


Getty Images/Fotog

5 A financial adviser put together an investment portfolio of \$200 000 for a client. The portfolio was spread over three investments – shares, a unit trust and a first mortgage. The annual return for the first year was \$16 240. The shares paid 6%, the unit trust 8% and the first mortgage 10%. The dollar return from the first mortgage was twice that from the shares. Write equations and solve them to find out how much was invested in each of the investments.

CHAPTER SUMMARY
THREE-DIMENSIONAL VECTORS

1

- In a three-dimensional coordinate system, the position of a point is defined by an **ordered triple** as in $P(x, y, z)$.
- The Cartesian (or rectangular) plane in three dimensions has three planes – the *xy*-, *yz*- and *xz*-planes. These planes partition space into eight **octants**.
- The **distance formula in three dimensions (3D distance formula)** states that for $P_1(x_1, y_1, z_1)$ and $P_2(x_1, y_1, z_2)$:

$$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$$

- The **midpoint** of the line segment from $P_1(x_1, y_1, z_1)$ to $P_2(x_2, y_2, z_2)$ is:

$$\left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}, \frac{z_1 + z_2}{2}\right)$$

- The **equation of a sphere** with centre $C(h, k, l)$ and radius, r, is:

$$(x - h)^2 + (y - k)^2 + (z - l)^2 = r^2$$

or $$x^2 + y^2 + z^2 + ax + by + cz + d = 0$$

- Properties of vectors.

  If $\mathbf{a} = (a_1, a_2, a_3)$ and $\mathbf{b} = (b_1, b_2, b_3)$ are vectors and c is a scalar:

  $\mathbf{a} + \mathbf{b} = (a_1 + b_1, a_2 + b_2, a_3 + b_3)$

  $c\mathbf{a} = (ca_1, ca_2, ca_3)$

  $0 = (0, 0, 0)$

  $-\mathbf{a} = -(a_1, a_2, a_3)$

  $\mathbf{a} - \mathbf{b} = (a_1 - b_1, a_2 - b_2, a_3 - b_3)$

- When vectors are added, their sum is called the **resultant** vector.
- A vector $\mathbf{a} = (a_1, a_2, a_3)$ can be represented in the three-dimensional Cartesian system by a directed line segment $\overrightarrow{QR}$ (or **QR**) with **initial point** (or head) $Q(x, y, z)$ and **terminal point** (or tail) $R(x + a_1, y + a_2, z + a_3)$.
- The vector **OA** from the origin, $O(0, 0, 0)$, to $A(a_1, a_2, a_3)$ is called the **position vector** of A. The lower case vector symbol is also used for the position vector, so $\mathbf{OA} = \mathbf{a}$.
- If $\mathbf{PQ} = \mathbf{RS}$, then **PQ** and **RS** have the same magnitude and direction. If $\mathbf{PQ} = c\mathbf{RS}$, where c is a scalar, then $\mathbf{PQ} \parallel \mathbf{RS}$.
- For any two points $P_1(x_1, y_1, z_1)$ and $P_2(x_2, y_2, z_2)$:

$$\mathbf{P_1P_2} = (x_2 - x_1, y_2 - y_1, z_2 - z_1)$$

- The magnitude of a three-dimensional vector is called the **norm** of the vector. For vector **a**, the norm is written $|\mathbf{a}|$. You can write $|\mathbf{a}|$ as a.
- If $\mathbf{a} = (a_1, a_2, a_3)$ and c is a scalar:

  $|\mathbf{a}| = \sqrt{a_1^2 + a_2^2 + a_3^2}$

  $|c\mathbf{a}| = c|\mathbf{a}|$

- A **unit vector** has magnitude 1. The unit vector in the same direction as a given vector **p** is written as $\hat{\mathbf{p}}$.

$$\hat{\mathbf{p}} = \frac{1}{|\mathbf{p}|}\mathbf{p}$$

- The unit vectors in space are:

  $\mathbf{i} = (1, 0, 0)$

  $\mathbf{j} = (0, 1, 0)$

  $\mathbf{k} = (0, 0, 1)$

- For any vector, $\mathbf{a} = (a_1, a_2, a_3)$:

$$\mathbf{a} = (a_1, a_2, a_3) = (a_1\mathbf{i}, a_2\mathbf{j}, a_3\mathbf{k})$$

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- The **polar form** of a vector has a magnitude r and two angles, θ and ϕ. The **azimuth angle** θ lies between 0° and 360° and is the angle from the x-axis to the projection of the vector onto the xy-plane (between x and a). The **altitude angle** ϕ lies between –90° and 90° and is the angle from the xy-plane to the vector (between a and r). The **spherical coordinates** have a **polar angle** ψ measured anticlockwise from the z-axis instead of the altitude angle. The polar angle lies between 0° and 180°.

- **3D vector forms**

  For the 3D vector $\mathbf{v} = (r, \theta, \phi) = (x, y, z)$:

  $x = r\cos(\phi)\cos(\theta)$ $\quad r^2 = x^2 + y^2 + z^2$

  $y = r\cos(\phi)\sin(\theta)$ $\quad \tan(\theta) = \dfrac{y}{x}$

  $z = r\sin(\phi)$ $\quad \sin(\phi) = \dfrac{z}{r}$ and

  $r = |\mathbf{v}| = \sqrt{x^2 + y^2 + z^2}$ $\quad \phi + \psi = 90°$

- If vectors **PQ** and **RS** have respective position vectors **a** and **b** and the angle between **a** and **b** is θ, the **scalar product** of **PQ** and **RS** is given by:

  $$\mathbf{PQ} \cdot \mathbf{RS} = \mathbf{a} \cdot \mathbf{b} = |\mathbf{a}||\mathbf{b}|\cos(\theta)$$

  $$\mathbf{a} \cdot \mathbf{b} = ab\cos(\theta)$$

- In component form, if $\mathbf{a} = (a_1, a_2, a_3)$ and $\mathbf{b} = (b_1, b_2, b_3)$

  $$\begin{aligned}\mathbf{a} \cdot \mathbf{b} &= (a_1, a_2, a_3) \cdot (b_1, b_2, b_3)\\ &= a_1b_1 + a_2b_2 + a_3b_3\end{aligned}$$

- The scalar product is also called the **inner product** or **dot product**. It is a scalar (real number), not a vector.

- Properties of the scalar product

  1. $\mathbf{a} \cdot \mathbf{b} = |\mathbf{a}|\,|\mathbf{b}|\cos(\theta)$, where θ is the angle between **a** and **b**.
  2. The scalar product is a real number, *not a vector*.
  3. The scalar product is commutative:
     $$\mathbf{a} \cdot \mathbf{b} = \mathbf{b} \cdot \mathbf{a}$$
  4. The scalar product is distributive over vector addition:
     $$\mathbf{a} \cdot (\mathbf{b} + \mathbf{c}) = \mathbf{a} \cdot \mathbf{b} + \mathbf{a} \cdot \mathbf{c}$$
  5. $m(\mathbf{a} \cdot \mathbf{b}) = (m\mathbf{a}) \cdot \mathbf{b} = \mathbf{a} \cdot (m\mathbf{b})\ \forall\ m \in \boldsymbol{R}$ and vectors **a**, **b**.
  6. $\mathbf{a} \cdot \mathbf{b} = 0 \Rightarrow \mathbf{a} = \mathbf{0}$ or $\mathbf{b} = \mathbf{0}$ or **a** and **b** are perpendicular.

- The angle between two vectors can be found using the scalar product.

  Since $\mathbf{a} \cdot \mathbf{b} = |\mathbf{a}||\mathbf{b}|\cos(\theta)$, $\cos(\theta) = \dfrac{\mathbf{a} \cdot \mathbf{b}}{|\mathbf{a}||\mathbf{b}|}$

- The magnitude of the **projection** of **a** on **b** can be calculated by the dot product of **a** and the unit vector in the direction of **b**.

  $$p = \mathbf{a} \cdot \hat{\mathbf{b}} \text{ or } \mathbf{p} = (\mathbf{a} \cdot \hat{\mathbf{b}})\,\hat{\mathbf{b}}$$

- The work done by a constant force **PQ** that results in a displacement **PR** is **PQ** · **PR**.

- The **vector product** of two vectors $\mathbf{v}_1$ and $\mathbf{v}_2$ with an angle of θ between them is given by

  $$\mathbf{v} = \mathbf{v}_1 \times \mathbf{v}_2, \text{ where } |\mathbf{v}| = |\mathbf{v}_1||\mathbf{v}_2|\sin(\theta).$$

  v is perpendicular to both $\mathbf{v}_1$ and $\mathbf{v}_2$ such that the three vectors $\mathbf{v}_1$, $\mathbf{v}_2$ and **v** form a right-handed system. The vector product is also called the **outer product** or **cross product**.

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- The vector product may be calculated from vectors in component form by using

  $(a_1, a_2, a_3) \times (b_1, b_2, b_3)$

  $= (a_2b_3 - a_3b_2, a_3b_1 - a_1b_3, a_1b_2 - a_2b_1)$

  or by using the rules: $\mathbf{i} \times \mathbf{i} = \mathbf{j} \times \mathbf{j} = \mathbf{k} \times \mathbf{k} = \mathbf{0}$,

  and $\mathbf{i} \times \mathbf{j} = \mathbf{k}$, $\mathbf{j} \times \mathbf{k} = \mathbf{i}$, $\mathbf{k} \times \mathbf{i} = \mathbf{j}$, $\mathbf{j} \times \mathbf{i} = -\mathbf{k}$, $\mathbf{k} \times \mathbf{j} = -\mathbf{i}$, $\mathbf{i} \times \mathbf{k} = -\mathbf{j}$.
- Properties of the vector product
  1. $|\mathbf{a} \times \mathbf{b}| = |\mathbf{a}|\,|\mathbf{b}|\,\mathbf{sin}\,(\theta)$, where θ is the angle between **a** and **b**.
  2. The vector product $\mathbf{a} \times \mathbf{b}$ is perpendicular to both **a** and **b**.
  3. $\mathbf{a} \times \mathbf{a} = \mathbf{0}$ for any vector **a**.
  4. The vector product is left- and right-distributive over vector addition.
  5. $\mathbf{a} \times \mathbf{b} = \mathbf{0} \Rightarrow \mathbf{a} = \mathbf{0}$ or $\mathbf{b} = \mathbf{0}$ or **a** and **b** are parallel.
  6. $\mathbf{a} \times \mathbf{b} = -\mathbf{b} \times \mathbf{a}\ \forall$ vectors **a**, **b**.
  7. $m(\mathbf{a} \times \mathbf{b}) = (m\mathbf{a}) \times \mathbf{b} = \mathbf{a} \times (m\mathbf{b})\ \forall\ m \in \boldsymbol{R}$ and vectors **a**, **b**.
- The vector product can also be calculated using the determinant-like method. If $\mathbf{a} = a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k}$ and $\mathbf{b} = b_1\mathbf{i} + b_2\mathbf{j} + b_3\mathbf{k}$, then use the arrangement

$$\begin{Bmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{Bmatrix}$$

  to obtain

$$\mathbf{a} \times \mathbf{b} = (a_2b_3 - a_3b_2)\mathbf{i} - (a_1b_3 - a_3b_1)\mathbf{j} + (a_1b_2 - a_2b_1)\mathbf{k}$$

- The magnitude of the vector product $\mathbf{a} \times \mathbf{b}$ equals the area of the parallelogram determined by **a** and **b**.
- Point O is always used for the origin, and the symbol **a** may be used instead of **OA** for the position vector of point A. It follows that $\mathbf{AB} = \mathbf{OB} - \mathbf{OA} = \mathbf{b} - \mathbf{a}$. The sum of a series of displacements 'end-to-end' can be replaced by the total displacement using just the first and last points, as in $\mathbf{AB} + \mathbf{BP} + \mathbf{PQ} + \mathbf{QE} = \mathbf{AE}$.
- The **vector equation** of the line through points A and B with position vectors **a** and **b** is:

  $\mathbf{p} = \mathbf{a} + t\mathrm{AB}$ (for some real number t)

  Alternatively, if **u** is any vector in the direction of the line AB:

  $\mathbf{p} = \mathbf{a} + t\mathbf{u}$

  For the line segment AB, the vector equation is $\mathbf{p} = \mathbf{a} + t\mathbf{AB}$ for $0 \le t \le 1$.
- The **Cartesian form** of the equation of a line in space passing through the points $A(a_1, a_2, a_3)$ and $B(b_1, b_2, b_3)$ is:

$$\frac{x - a_1}{b_1 - a_1} = \frac{y - a_2}{b_2 - a_2} = \frac{z - a_3}{b_3 - a_3}$$

- If the vector equation of a line is $\mathbf{p} = \mathbf{a} + t\mathbf{u}$, where **u** is a vector in the direction of the line, and if $\mathbf{a} = a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k}$ and $\mathbf{u} = u_1\mathbf{i} + u_2\mathbf{j} + u_3\mathbf{k}$, then the **Cartesian form** of the equation is:

$$\frac{x - a_1}{u_1} = \frac{y - a_2}{u_2} = \frac{z - a_3}{u_3}$$

- The **Cartesian equation of a plane** through $P(x_1, y_1, z_1)$, with normal vector $\mathbf{n} = (a_1, a_2, a_3)$ is:

$$a_1(x - x_1) + a_2(y - y_1) + a_3(z - z_1) = 0$$

- The **vector equation of the plane** through $P(x, y, z)$ and $A(a_1, a_2, a_3)$, with position vectors **a** and **p** respectively and with normal vector **n** is:

$$\mathbf{n} \cdot (\mathbf{p} - \mathbf{a}) = 0$$

  This equation may also be written as:

  $\mathbf{n} \cdot \mathbf{p} = \mathbf{n} \cdot \mathbf{a} = k$ or $\mathbf{p} \cdot \mathbf{n} = \mathbf{a} \cdot \mathbf{n} = k$, where k is a constant.

- The graph of the equation $lx + my + nz = k$ is a plane with normal $\mathbf{n} = (l, m, n)$.
- In the **parametric form** of a line, the coordinates are expressed as functions of the parameter. If one or more of these coordinate functions is not a linear function, the line will be curved.
- A **linear equation** is one that can be written with no powers, roots, reciprocals, and so on—for example, $5x - y + 4z = 8$. A **system** of equations shows the relationships between a group of variables.
- **Gaussian elimination** steps for multiple equations

  1 Choose which variable to eliminate.

  2 Put the equation with the simplest coefficient of this variable first in the list.

  3 For each of the other equations, multiply the chosen equation by a suitable number and then add to eliminate the chosen variable.

  4 Repeat steps 1–3 with the new equations until an equation with a single variable is obtained, if possible. Obtain its value.

  5 Progressively substitute to find the values of all the variables.
- There are three cases that occur for simultaneous linear equations.

  A **unique solution** corresponds to planes that intersect at a single point.

  Consistent equations have at least one solution.

  Inconsistent equations correspond to cases involving at least a pair of parallel planes or planes that intersect in parallel lines.

  Dependent equations have an equation that can be written as a combination of the others. This corresponds to planes that intersect in a straight line.
- Simultaneous equations with many variables are normally solved using approximate matrix methods to obtain 'good enough' solutions. CAS calculators may also be used.

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CHAPTER REVIEW THREE-DIMENSIONAL VECTORS

1

Multiple choice

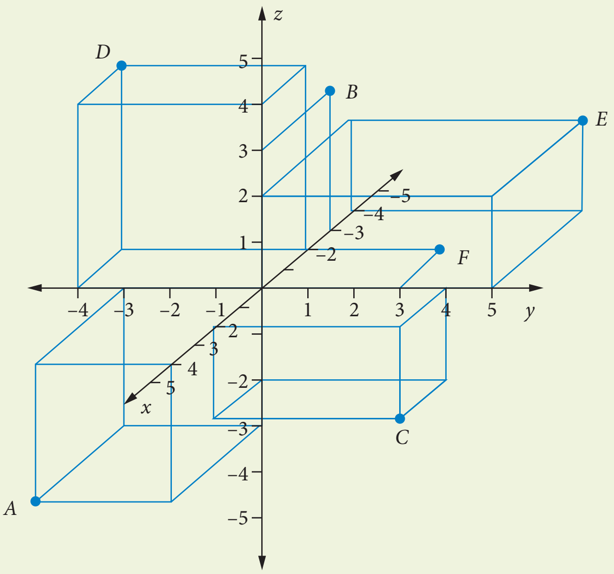
1 Example 1 $P(-1, 4, -3)$ and which of the following points are on a line parallel with the y-axis?

A $(-1, 3, 2)$ B $(-1, -5, -3)$ C $(2, 4, -3)$ D $(-1, 4, 3)$ E $(1, 4, 3)$

2 Example 1 The distance between $M(p, q, r)$ and $N(p, q, s)$ is:

A $|s-p|$ B $|p-r|$ C $|r-s|$ D $|r-q|$ E $|q-s|$

Use the diagram below to answer questions **3–4**.



3 Example 1 The point lying on the xz-plane is:

A A B B C C D D E E

4 Example 1 The point F is best described as lying on the:

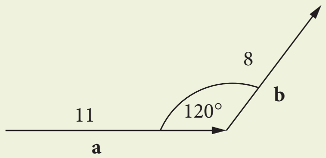
A xy-plane B y-plane C xz-plane
D x-plane E yz-plane

5 Example 7 The dot product of $-5\mathbf{i} + 2\mathbf{j} - \mathbf{k}$ and $3\mathbf{i} - 4\mathbf{j} - 7\mathbf{k}$ is:

A -30 B -16 C -12 D 0 E 16

6 Example 7 Given the two vectors shown on the right, $\mathbf{a} \cdot \mathbf{b}$ is closest to:

A -44 B -35.4 C 44
D 76.2 E 88



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7 Example 10 Given the vectors **p** and **q**, $\mathbf{p} \times \mathbf{q}$ is:

A parallel to **p** and **q**

B parallel to **q** and perpendicular to **p**

C perpendicular to **p** and **q**

D parallel to **p** and perpendicular to **q**

E None of the above is true.

8 Example 10 Given $\mathbf{a} = \mathbf{i} + 2\mathbf{j} + 3\mathbf{k}$, $\mathbf{b} = 3\mathbf{i} + 2\mathbf{j} + \mathbf{k}$ and $\mathbf{c} = -\mathbf{i} - 2\mathbf{j} - 3\mathbf{k}$, $\mathbf{a} \times \mathbf{b}$ is equal to:

A $\mathbf{a} \times \mathbf{c}$ B $\mathbf{c} \times \mathbf{b}$ C $\mathbf{b} \times \mathbf{a}$ D $\mathbf{c} \times \mathbf{a}$ E $-(\mathbf{c} \times \mathbf{b})$

9 Example 10 Given $A(-9, 7, 2)$ and $B(-4, 3, 1)$, $\mathbf{a} \times \mathbf{b}$ is equal to:

A $\mathbf{i} + \mathbf{j} + \mathbf{k}$ B $\mathbf{0}$ C $\mathbf{AB}$ D $\mathbf{i} - 17\mathbf{j} + \mathbf{k}$ E $\mathbf{i} - \mathbf{j} + \mathbf{k}$

Short answer

10 Example 2 Given $A(1, -5, 3)$, $B(-4, 3, -2)$ and $C(5, 0, -6)$, find the straight-line distances and midpoints of the line segments:

a AB b BC c CA

11 Example 3 Find the equation of a sphere with radius 5 and centre $(3, -4, -2)$.

12 Example 3 Find the radius and centre of the sphere

$$x^2 + y^2 + z^2 - 6x + 2y - 4z + 5 = 0$$

13 Example 4 Given $\mathbf{p} = -3\mathbf{i} + \mathbf{j} + 2\mathbf{k}$, $\mathbf{q} = 4\mathbf{i} - 2\mathbf{j} + \mathbf{k}$, A $(4, 2, -3)$ and $B = (5, -1, 6)$:

a find $\mathbf{AB}$ and $|\mathbf{AB}|$

b find $\hat{\mathbf{p}}$

14 Example 4 Given $\mathbf{m} = (-6, 4, -2)$, $\mathbf{z} = (0, 3, -4)$, $\mathbf{d} = (1, 2, 5)$ and $\mathbf{e} = (3, -2, -1)$, find:

a $\mathbf{m} + \mathbf{e}$ b $\mathbf{d} - 2\mathbf{z}$ c $\mathbf{m} - \mathbf{d}$ d $3\mathbf{e} - 4\mathbf{z}$ e $4\mathbf{d} - 3\mathbf{m}$

15 Examples 4, 6 a Convert $(3, -7)$ and $(3, -2, 4)$ to polar form.

b Convert $(8, 210°)$ and $(20, 160°, -36°)$ to component form.

c Find the norms of $(-5, 12)$ and $(18, 10, -15)$.

16 Example 5 Write the following as linear combinations of **i**, **j** and **k**.

a $(5, -7)$ b $(25, 330°)$ c $(7, 8, -2)$ d $(30, 60°, 30°)$

17 Example 5 Express the following vectors as linear combinations of **i**, **j** and **k**, and state the norm of each vector.

a $\begin{bmatrix} 3 \\ -1 \\ 4 \end{bmatrix}$ b $\begin{bmatrix} -3 \\ -5 \\ 2 \end{bmatrix}$ c $(1, -3, 0)$ d $(2, 5, -3)$

e Magnitude 5, azimuth 240°, altitude 43°

f Magnitude 35.7, azimuth 63°, altitude –67°

18 Example 6 Find the resultant of vectors **p** and **q**, where **p** has magnitude 25, azimuth 120° and altitude –30°, while **q** has magnitude 30, azimuth 60° and altitude 45°.

19 Example 7 Find the scalar products of the following.

a Vectors with magnitudes 5 and 8 at an angle of 75°

b (3, –5) and (6, 4)

c (0, –3, 7) and (3, 5, 2)

20 Example 7 Find the vector products of the following.

a Vectors with magnitudes 8 and 10 at an angle of 63°

b (14, 62°) and (25, 140°)

c (3, 5, –1) and (–2, 3, 4)

d $2\mathbf{i} - 3\mathbf{j} + \mathbf{k}$ and $-3\mathbf{i} + 2\mathbf{k}$

21 Example 8 Given $\mathbf{a} = 3\mathbf{i} - 2\mathbf{j} + \mathbf{k}$, $\mathbf{b} = 5\mathbf{i} + 3\mathbf{j} - 4\mathbf{k}$ and $\mathbf{c} = \mathbf{i} - \mathbf{j} + 3\mathbf{k}$, find:

a $\mathbf{a} \cdot \mathbf{c}$ b $\mathbf{b} \cdot \mathbf{c}$ c $\mathbf{a} + \mathbf{b}$ d $(\mathbf{a} + \mathbf{b}) \cdot \mathbf{c}$ e $\mathbf{c} \cdot \mathbf{b}$

22 Example 8 Given $\mathbf{a} = 2\mathbf{i} + \mathbf{j} - 3\mathbf{k}$, $\mathbf{b} = \mathbf{i} - 2\mathbf{j} + 4\mathbf{k}$, $\mathbf{c} = 3\mathbf{i} - \mathbf{j} + 2\mathbf{k}$, and $\mathbf{d} = 2\mathbf{i} - 2\mathbf{j} - 3\mathbf{k}$, find:

a the projection of **a** on **b** using the scalar product $\mathbf{a} \cdot \mathbf{b}$

b the projection of **c** on **d** using unit vectors

c the angle between **a** and **c**

23 Example 9 Find the amount of work done when a constant force $\mathbf{a} = 2\mathbf{i} - 3\mathbf{j} + 5\mathbf{k}$ N moves its point of application from $P(3, 2, -2)$ to $Q(4, -4, 7)$, if displacement is measured in metres.

24 Find the vector product of **a** and **b** if:

a $|\mathbf{a}| = 7$, $|\mathbf{b}| = 15$ and the angle between **a** and **b** is 48°

b $\mathbf{a} = (3, -2, 5)$ and $\mathbf{b} = (-3, 4, 7)$

25 Example 10 Given $\mathbf{a} = \begin{bmatrix} 2 \\ -3 \\ -2 \end{bmatrix}$, $\mathbf{b} = \begin{bmatrix} -2 \\ -2 \\ -4 \end{bmatrix}$, $\mathbf{c} = \begin{bmatrix} 4 \\ 8 \\ -5 \end{bmatrix}$, and $\mathbf{d} = \begin{bmatrix} 0 \\ -1 \\ 4 \end{bmatrix}$, find:

a $\mathbf{a} \times (\mathbf{b} \times \mathbf{c})$ b $\mathbf{a} \cdot \mathbf{b} \times \mathbf{c}$ c $(\mathbf{a} \times \mathbf{b}) \times \mathbf{c}$ d $\mathbf{c} \times \mathbf{d}$

e $\mathbf{d} \times \mathbf{c}$ f $\mathbf{b} \times (\mathbf{b} \times \mathbf{c})$ g $(\mathbf{b} \times \mathbf{b}) \times \mathbf{c}$ h $\mathbf{d} \times \mathbf{d}$

26 Example 11 Use the vector product to find the area of a triangle with vertices (1, –3, 2), (2, 1, 3) and (–2, –1, 2).

27 Example 12 Find a vector normal to the plane containing points (1, –2, 3), (4, 5, –2) and (–1, –2, 1).

28 Examples 16, 17 Find the vector and Cartesian equations of the line through the points (3, –7, 4) and (–5, 8, 2).

29 Examples 16, 17 A line passes through the points (2, –1, 5) and (3, 6, –4).

a Find the vector and Cartesian equations of the line.

b What is the vector equation of the line segment between the points?

c Does the point (0, –15, 9) lie on the line?

30 Example 18 Find the vector equation of the line that passes through the point $P(-1, 2, 1)$ that is parallel to the line passing through $A(2, 3, -1)$ and $B(1, -2, 3)$ and show that $Q(-3, -8, 9)$ lies on the line.

31 Example 18 Two lines have vector equations $\mathbf{p} = \mathbf{i} - 2\mathbf{k} + t(2\mathbf{i} + 3\mathbf{j})$ and $\mathbf{q} = 3\mathbf{i} + 6\mathbf{j} - \mathbf{k} + s(2\mathbf{i} - \mathbf{k})$.
Show that the lines intersect and find their point of intersection.

32 Example 19 Consider the plane that passes through $Q(1, -1, 1)$ with normal $\mathbf{n} = \mathbf{i} + \mathbf{j} + \mathbf{k}$.
- **a** Find the Cartesian equation of a plane.
- **b** Does the point $(-1, 2, 3)$ lie on the plane?
- **c** Find an expression for points on the plane inside the circle of radius 5 and centre Q.

33 Example 20 Find the equation of the plane that passes through $P(6, 5, -4)$ and which is parallel to the plane $5x - 2y + 3z + 10 = 0$.

34 Example 21 Find the equation of the plane containing the points:
- **a** $(1, 0, 1)$, $(2, 4, 6)$ and $(1, 2, -1)$
- **b** $(1, -2, -3)$, $(4, -4, 4)$ and $(3, 2, -3)$

35 Examples 22–24 Solve the following systems of equations by Gaussian elimination.

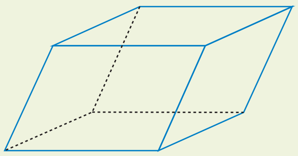
a
$$\begin{aligned} 5x + 21y + 2z &= 5 \\ 4x + 27y + z &= 5 \\ 2x + 12y + z &= 4 \end{aligned}$$

b
$$\begin{aligned} 4a - 6b + 3c &= 0 \\ a - 2b + c &= -6 \\ 2a - 2b + c &= -4 \end{aligned}$$

c
$$\begin{aligned} 4p - 12q - 4r &= 0 \\ 2p - 5q - r &= 3 \\ 8q - 3p + 2r &= -3 \end{aligned}$$

Application

36 Find the volume of a parallelepiped with vertices $(3, 5, 7)$, $(-2, 1, 9)$, $(4, 2, 5)$, $(9, 6, 3)$, $(1, 0, 2)$, $(-4, -4, 4)$, $(2, -3, 0)$ and $(7, 1, -2)$.



37 Show that for any set of points A, B, C, D, E:
$(2\mathbf{AB} - \mathbf{AC}) \times \mathbf{DE} + \mathbf{BC} \times \mathbf{DE} + \mathbf{AB} \times (\mathbf{CD} - \mathbf{CE}) = \mathbf{0}$

38 Use vector methods to prove that the angle bisector of the unequal angle of an isosceles triangle is also the altitude and median.

39 Find counterexamples to show that $\mathbf{a} \cdot \mathbf{b}$ and $\mathbf{a} \times \mathbf{b}$ can be zero without $\mathbf{a}$ or $\mathbf{b}$ being zero vectors.

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40 A plumber regularly buys three sizes of plastic drainpipe in 6 m lengths: 40 mm, 50 mm and 100 mm. In January, his supplier's account included the following:

9 Jan:	3×40 mm, 10×50 mm, 4×100 mm	\$475.80
17 Jan:	4×40 mm, 2×50 mm, 1×100 mm	\$158.20
24 Jan:	1×40 mm, 5×50 mm, 2×100 mm	\$230.20

Set up equations and use Gaussian elimination to find the price of each type of pipe.

Getty Images/Bloomberg

41 The median of a triangle is a vector from a vertex to the midpoint of the opposite side. Show that the sum of the medians of a triangle = 0.

42 Show that the midpoint of the hypotenuse of a right-angled triangle is equidistant from the vertices of the triangle.

43 If *ABCD* is a parallelogram and *M* is the midpoint of *AC*, prove that *M* is also the midpoint of *BD*.

Qz Practice quiz